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Crack–Inclusion Interaction: A Review

by Christopher S. Meyer

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14. ABSTRACT

This report reviews the fundamentals of the interaction between a crack and an inclusion in engineering materials. After a discussion of inclusions and cracks and a brief historical review of work that lead to the solution, the report works through the solution for a crack-inclusion problem in fracture mechanics and then discusses some relevant results. Specifically, this report reviews the foundational dislocation Green's function and integral equation solution to an elliptical inclusion and a straight crack in an infinite elastic medium. This report is intended as a detailed introduction to the crack-inclusion interaction problem in fracture mechanics, so it is not an exhaustive review of literature on the subject and it does not include much of the historical references or any recent developments in this research area. This report aims to provide the attentive reader who is familiar with basic linear elastic fracture mechanics with enough depth of knowledge in the crack-inclusion interaction problem to understand further research on the problem.

15. SUBJECT TERMS

fracture mechanics, inclusion, crack, dislocation, Erdogan, Dundurs, integral equation, Green's function

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1. Introduction

Fracture in engineering materials, particularly metals, often initiates at inclusions and voids (*1*). Figure 1 is a scanning electron micrograph of the fracture surface from a failed type 431 stainless steel bolt from a helicopter. The dark, geometric shapes on the fracture surface are grains, and the appearance of the fracture surface is typical of brittle, intergranular fracture. The lighter-colored amorphous objects on the fracture surface are inclusions. As figure 1 illustrates, cracks often interact with inclusions and voids. This interaction is a real-world problem and much research has been conducted since the early 1960s in the fracture mechanics area of the crack-inclusion interaction. Developing an understanding of the crack-inclusion interaction is important to mechanical design, life cycle predictions, failure analysis, and many other engineering considerations. This report reviews some of the foundational work in the crack-inclusion interaction of fracture mechanics.

There are many types of inclusions in engineering materials. In metals, we find casting defects, dislocations, different phases, impurities, and voids. For example, silicon carbide particles can be inclusions in aluminum. Ceramics often contain various phases such as graphite inclusions in diamond. In composites, we may consider the reinforcing materials as inclusions within a matrix, for example, carbon fibers in an epoxy matrix. In building materials, we see sand and gravel as inclusions in cement and aggregate of various sizes in concrete and bricks, and adobe is full of sand inclusions within a clay matrix. Building materials are also riddled with voids and porosity. Polymers contain varying degrees of crystallinity and varying lengths of polymer chains, and these too may be considered inclusions. Even biological materials contain inclusions—for example, voids in the foam-like structures of bones and bird feathers may be thought of as inclusions. Perhaps we could say any inhomogeneity in engineering materials is an inclusion. Since we know there is no truly homogeneous material, all materials may be said to have inclusions, so the importance of understanding the interaction between a crack and an inclusion becomes clear. But to simplify our considerations, we may think of inclusions as either different phases within a material or void, which is the absence of material. In accordance with this idea, we consider inclusions to be divided into these two categories: voids and phases.

Figure 2 is a micrograph of a sample of poorly welded titanium (Ti-6Al-4V). To avoid oxidation, the alloy must be welded in an inert atmosphere and so typically argon is flooded over the titanium as it is welded. During the welding process, the grains of metal melt, recrystallize, and grow. If the process is poorly controlled, voids may develop as the inert gas is trapped within the

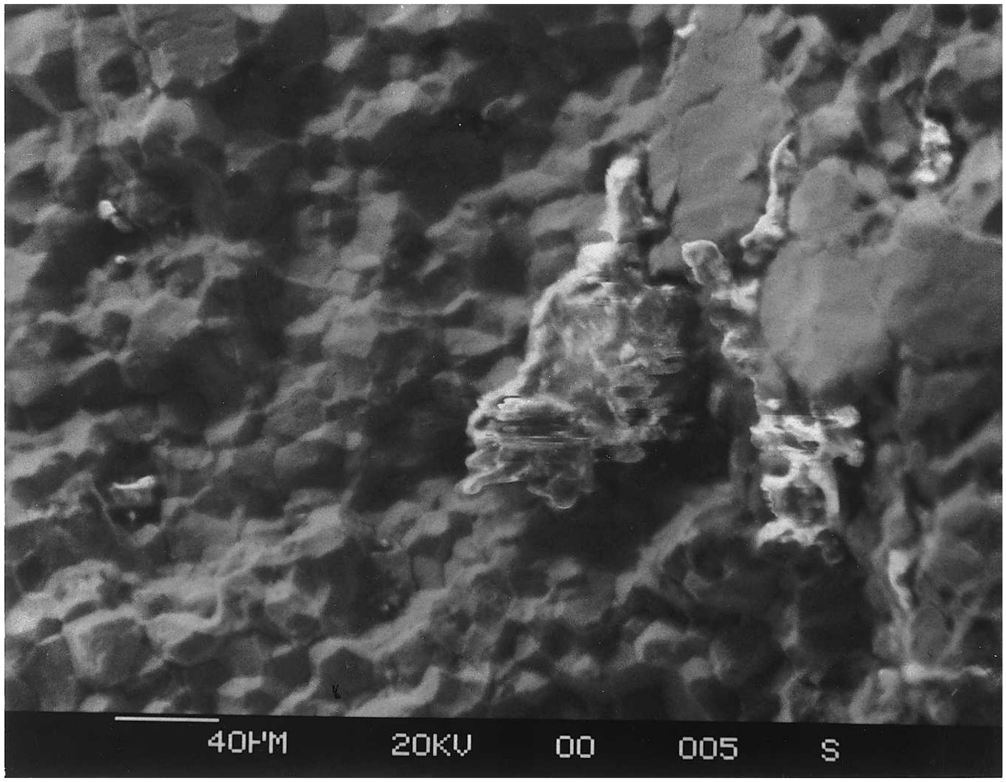

Figure 1. Micrograph illustrating interaction between fracture surface and inclusions.

structure or bubbles up to the surface. Figure 3 shows a region of heat-treated steel that has undergone a martensitic transformation. The pre-heat-treated microstructure of steels is often largely composed of cementite (Fe_3C) and ferrite (α phase iron). Heat treatment of steels causes phase transformation to austenite (γ phase iron), and if the steel is quenched, the austenite will transform into martensite. Martensite grains are much smaller than austenite grains, and this contributes to the increased strength of the martensite structure (see the Hall-Petch equation, which relates the strength of a metal to the reciprocal of the square root of its grain size, for example, Reed-Hill and Abbaschian, (*1*), which has an excellent discussion on steel microstructures and strength). The martensitic transformation also causes an increase in volume, and this is why manufactured steel parts are often machined to near net shape prior to heat treating and then machined to the final shape after heat treatment. However, sometimes the martensitic transformation may remain incomplete such that there is a small amount of retained austenite. The bright white spots in figure 3 are retained austenite. Obviously, these two phases of steel will have different elastic properties. How will a crack behave when it approaches an inclusion that is less stiff than the surrounding material? How will a crack behave if it approaches an inclusion that is stiffer than the surrounding material?

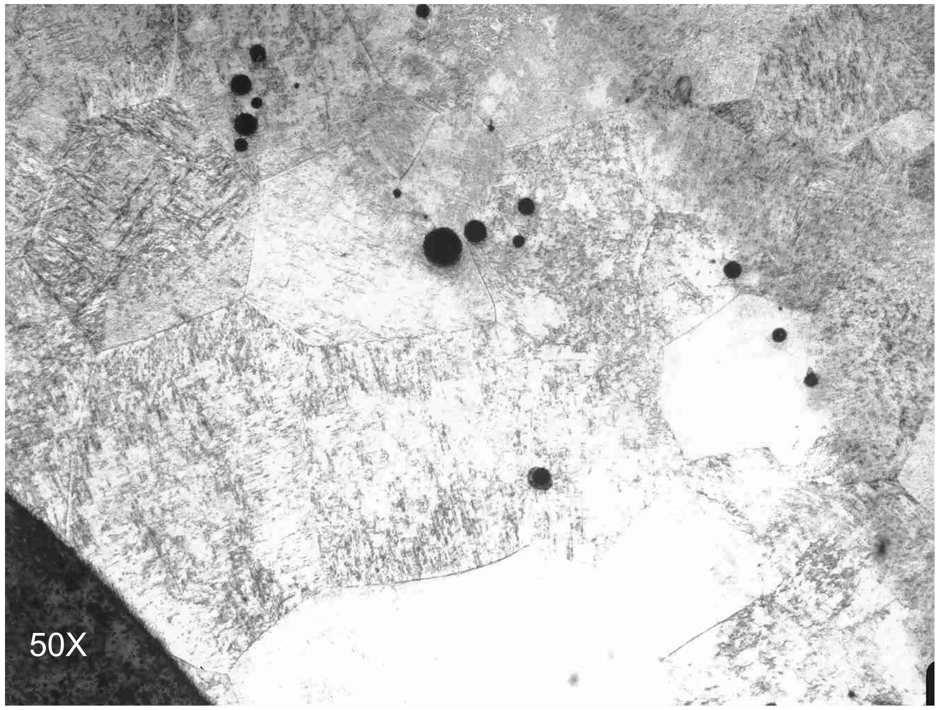

Figure 2. Micrograph of voids in titanium.

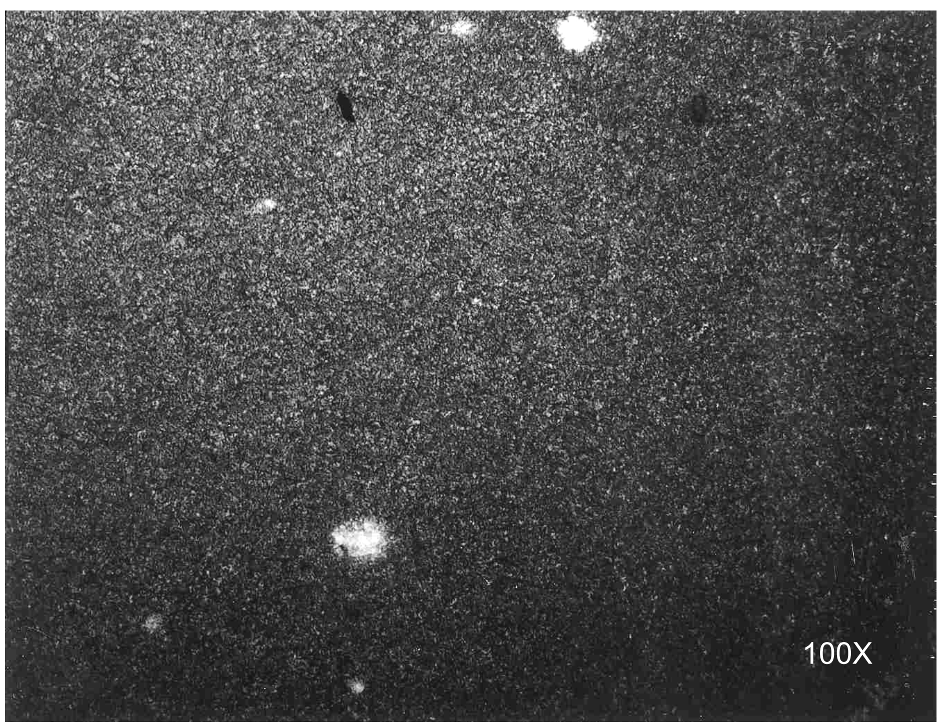

Figure 3. Micrograph of retained austenite (white) in martensitic steel.

Figure 4 shows small circular carbide inclusions in steel, as seen in a cross section perpendicular to the primary fracture surface. The secondary crack on the left is approaching a carbide inclusion and the secondary crack on the right has traveled around a carbide inclusion. A crack may pass near an inclusion, perhaps attracted to the inclusion; or a crack may move away from an inclusion, perhaps repelled by the inclusion; or a crack may cross the boundary into and perhaps propagate through an inclusion. Additionally, we might consider whether the inclusion is perfectly bonded to the matrix, partially bonded to the matrix, or not bonded to the matrix. Each of these considerations affects the result. In this study, we assume that a crack passes near an inclusion, and the inclusion is perfectly bonded to the surrounding matrix.

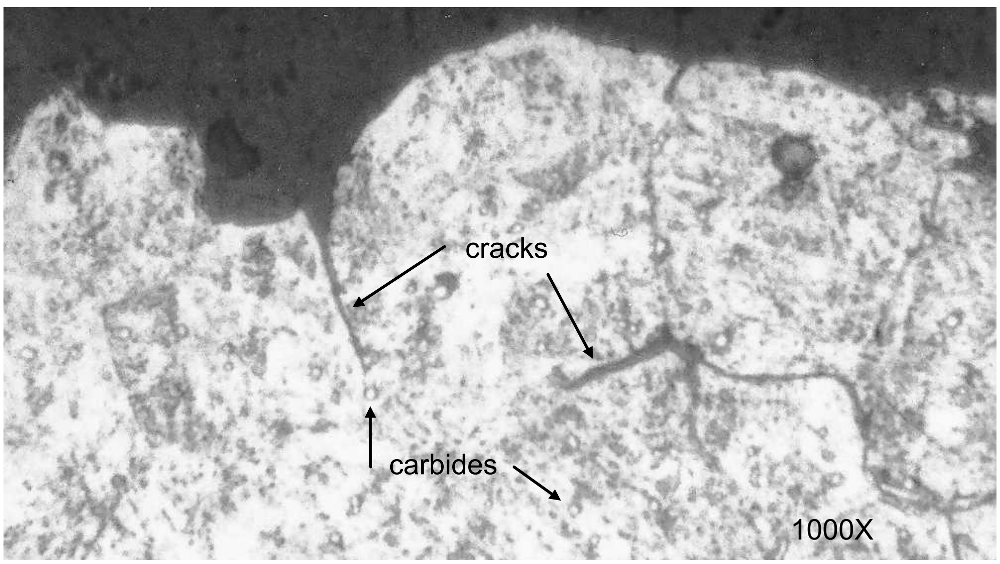

Figure 4. Micrograph of carbide inclusions in fractured steel.

2. Inclusions and Cracks

According to King (*2*), renowned scientist John D. Eshelby was a "pure theoretician" who would rather reason his way to the answer than conduct experiments. By 1961, Eshelby had authored seminal works on the mechanics of inclusions (*3, 4*). Famous for his thought experiments, Eshelby determined the stress field around an ellipsoidal inclusion using the superposition of two linear elastic problems solved using a Green's function approach (*5*). He imagined a homogeneous linear elastic solid volume, for example, figure 5a, with some known elastic constants, say E_{ijkl}. Within this semi-infinite volume he imagined a smaller, ellipsoidal volume—the inclusion. The ellipsoidal volume experiences a uniform, inelastic deformation, for

example, a martensitic transformation as discussed earlier, which causes the inclusion to grow in volume. Since the inclusion is confined by the surrounding matrix, the deformation puts the inclusion and the matrix into some state of stress. Eshelby imagined first removing the inclusion from the matrix, as seen in figure 5b, where the dotted line is the shape of the inclusion when it is confined by the matrix. Once removed from that confinement, the now free inclusion will relieve the deformation and seek equilibrium. Then with no force on the matrix volume, V_M, or the inclusion volume, V_I, the inclusion will assume a uniform strain, which is known as the eigenstrain, ε^*_{ij}, that is, the strain at zero stress. Then the strain, stress, and displacement in the matrix and inclusion are given by equations 1 and 2, respectively.

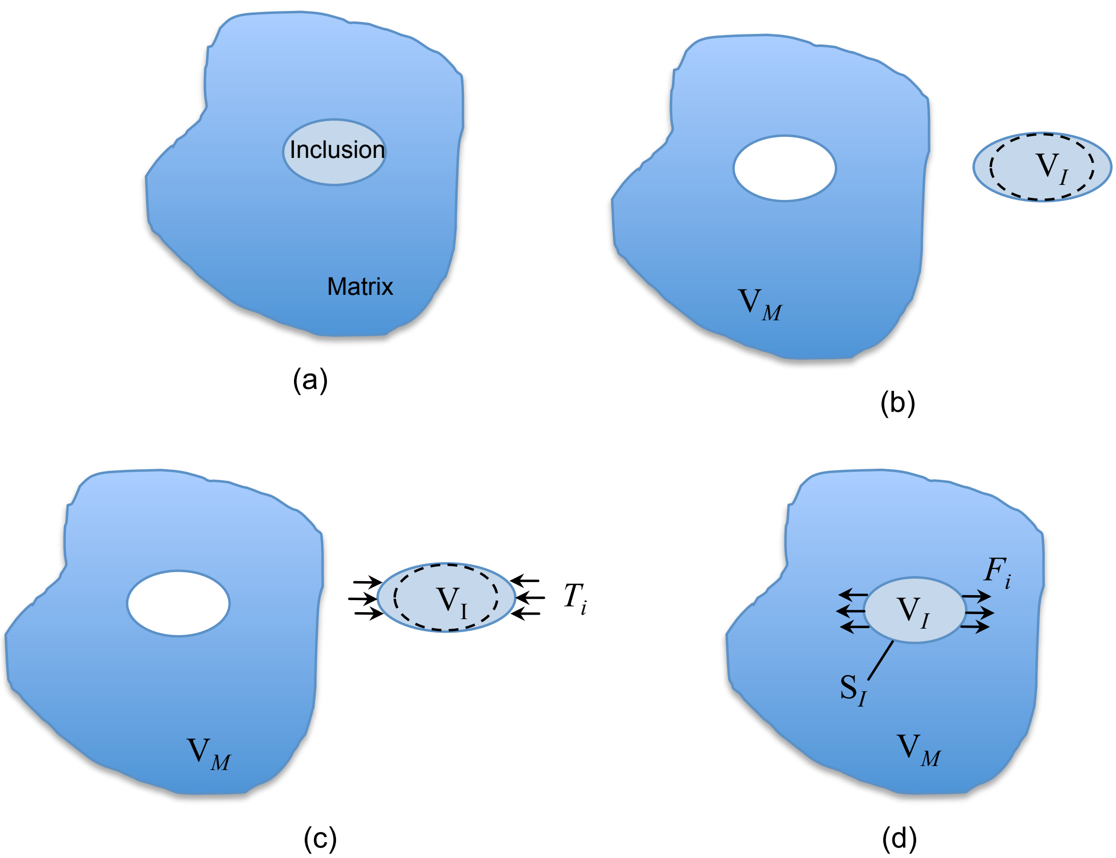

Figure 5. Eshelby's inclusion analysis.

$$
\begin{aligned}
\varepsilon_{ij} &= 0 \, , \\
\sigma_{ij} &= 0 \, , \\
u_i &= 0 \, .
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\varepsilon_{ij} &= \varepsilon^*_{ij} \ , \\
\sigma_{ij} &= 0 \ , \\
u_i &= \varepsilon^*_{ij} x_j \ .
\end{aligned}
\tag{2}
$$

Next, Eshelby imagined applying a traction, T_j, to the surface of the inclusion sufficient to force the inclusion into its original shape, the dotted line in figure 5c. This traction induces an elastic strain in the inclusion that exactly cancels the eigenstrain, that is, $\varepsilon^{el}_{ij} = -\varepsilon^*_{ij}$. The traction on the inclusion surface is given by equation 3, where σ^*_{ij} is the stress computed from the eigenstrain. Then the strain, stress, and displacement in the matrix are still given by equation 1, but the strain, stress, and displacement in the inclusion are now given by equation 4.

$$
T_j = \sigma_{ij} n_j = -\sigma^*_{ij} n_j \tag{3}
$$

$$
\begin{aligned}
\varepsilon_{ij} &= \varepsilon^{el}_{ij} + \varepsilon^*_{ij} = 0 \ , \\
\sigma_{ij} = E_{ijkl}\varepsilon^{el}_{ij} &= -E_{ijkl}\varepsilon^*_{ij} = -\sigma^*_{ij} \ , \\
u_i &= 0 \ .
\end{aligned}
\tag{4}
$$

With the inclusion deformed back to its original shape by the application of traction T_j, Eshelby imagined that the inclusion was placed back into the matrix. There is no change in the strain, stress, and displacement of the inclusion since the traction T_j is still applied; that is, the state of stress of the matrix is still given by equation 1 and the state of stress in the inclusion is still given by equation 4. But, if traction T_j were removed from the inclusion, the system would return to its original strain, stress, and displacement state, which is not known. So instead of removing the known traction T_j that was applied, Eshelby applied a canceling force $F_j = -T_j$ to the internal surface of the matrix, the surface that was exposed when the inclusion was removed, surface S_I in figure 5d. Then Eshelby was able to quantify the displacement field of the elastic body, called the constrained displacement field, in terms of the applied force F_j on the surface S_I, using the Green's function of the elastic body, that is, the matrix.

From equation 3 and $F_j = -T_j$ we have

$$F_j = -T_j = \sigma^*_{jk} n_k \ . \tag{5}$$

With the Green's function, G_{ij}, for the elastic body, the constrained displacement, u^c_i, is given by

$$u^c_i(x_i) = \int_{S_I} F_j(x'_i) G_{ij}(x_i, x'_i) dS(x'_i) = \int_{S_I} \sigma^*_{jk} n_k(x'_i) G_{ij}(x_i, x'_i) dS(x'_i) \ . \tag{6}$$

We know that strain is the derivative of displacement (the displacement gradient), so the strain in the elastic body is given by

$$u^c_{i,j}(x_i) = \int_{S_I} \sigma^*_{lk} n_k(x'_i) G_{il,j}(x_i, x'_i) dS(x'_i) \ , \tag{7}$$

$$\varepsilon^c_{ij}(x_i) = \frac{1}{2}(u^c_{i,j} + u^c_{j,i}) = \frac{1}{2} \int_{S_I} \sigma^*_{lk} n_k(x'_i)[G_{il,j}(x_i, x'_i) + G_{jl,i}(x_i, x'_i)] dS(x'_i) \ , \tag{8}$$

and stress in the elastic body due to the constrained displacement field is given by

$$\sigma^c_{ij}(x_i) = E_{ijkl} \varepsilon^c_{kl}(x_i) \ . \tag{9}$$

Finally, we can solve for the state of stress in the inclusion from the state of stress in the surrounding matrix. The displacement and strain will be the same since the inclusion is assumed to be perfectly bonded to the surrounding matrix. The stress in the inclusion will be equal to the stress in the surrounding matrix minus the stress in the inclusion due to the applied traction. Therefore, the strain, stress, and displacement in the matrix is given by equation 10 and the strain, stress, and displacement in the inclusion is given by equation 11.

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon^c_{ij} \ , \\ \sigma_{ij} &= \sigma^c_{ij} \ , \\ u_i &= u^c_i \ . \end{aligned} \tag{10}$$

$$
\begin{aligned}
\varepsilon_{ij} &= \varepsilon^c_{ij} \ , \\
\sigma_{ij} = \sigma^c_{ij} - \sigma^*_{ij} &= E_{ijkl}(\varepsilon^c_{kl} - \varepsilon^*_{kl}) \ , \\
u_i &= u^c_i \ .
\end{aligned}
\tag{11}
$$

We now have a linear elastic solution for the strain, stress, and displacement both inside and outside of an elliptical inclusion in an infinite elastic body. Further mathematical work involving Eshelby's tensor, S_{ijkl}, which relates the constrained strain to the eigenstrain as $\varepsilon^c_{ij} = S_{ijkl}\varepsilon^*_{kl}$, may be needed to identify explicit expressions, but this is discussed in more detail by Weinberger et al. (*5*). What we have learned is that all we have to do to obtain numerical values for these states of stress is identify the Green's function for the elastic body.

But we are interested in how a crack interacts with an inclusion. So let us consider how the stress field is affected by the presence of a crack. If we know the solution for the stress field in an infinite elastic body due to the presence of a crack, then, as in figure 6, we can superpose that solution on the solution we just found for the stress field of an infinite elastic body with an inclusion. Therefore, if we want to know how a crack will interact with an inclusion, all we need is the stress solution for a cracked infinite elastic body and the Green's function for an infinite elastic body with an inclusion.

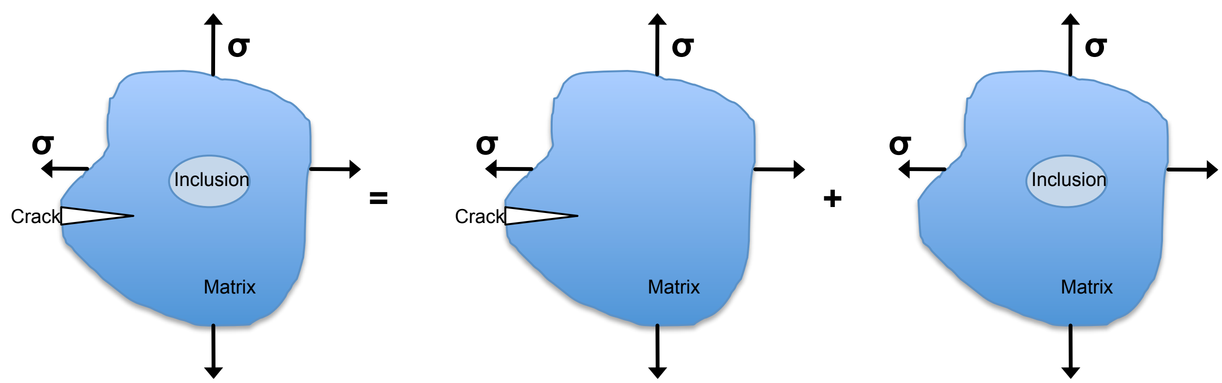

Figure 6. Superposition of solutions from a cracked elastic body and an elastic body with an inclusion.

3. Dislocations, Green's Functions, and Superposition

3.1 Dislocations and Inclusions

A dislocation is a line-shaped defect in a crystal lattice where there is mismatch in the crystal structure. Edge dislocations are formed by deformation of a body; they lie perpendicular to their Burgers vector and move in the direction of the Burgers vector through the body under load. For a much more detailed discussion on dislocations and Burgers vectors, see, for example, Reed-Hill and Abbaschian (*1*) or Tadmor and Miller (*6*).

In 1964, John Dundurs and Toshio Mura, both professors at Northwestern University, described a solution for an arbitrarily oriented single edge dislocation interacting with an inclusion in an elastic body (*7*). They found the solution by assuming the Airy stress potential and solving backwards to determine the terms by intuition and trial and error (*8*).

Dundurs and Mura provided the first solution in linear elasticity for an edge dislocation in an infinite elastic medium containing a circular inclusion, which has elastic properties different from the surrounding matrix (*7*). They assumed a perfect bond between matrix and inclusion so that the displacement and stress fields are the same for both matrix and inclusion at the boundary between the matrix and the inclusion. Further, they assumed there is a singularity at the edge dislocation, but no singularity anywhere else in the matrix or inclusion. Also, any closed contour integral that does not contain the center of the dislocation is required to equal zero, but any closed contour integral that does contain the center of the dislocation is required to equal the Burgers vector of the dislocation, b. Finally, the stress at infinity is assumed to vanish.

The complex potential method of solution for a dislocation interacting with an inclusion is beyond the scope of this report. Dundurs and Santare (*8*) provide an excellent discussion of the topic and a review of the literature.

3.2 Dislocations and Green's Functions

Dundurs and Santare (*8*) provide a concise discussion on using a "concentrated unit of inhomogeneity" as a forcing function in the solution to a boundary value problem, which they attribute to Stakgold (*9*) who first wrote of this possibility in 1967. Known as a Green's function, Dundurs and Santare (*8*) discuss that the solution to this concentrated unit of inhomogeneity, the Green's function, may be used to find the solution for a general boundary value problem in terms

of a continuous forcing function. The boundary value problem they discuss is the displacement, strain, and stress fields and the corresponding boundary conditions found in elasticity problems. Since the forcing function is provided by a concentrated unit of inhomogeneity, we may consider, for example, a point load in an elasticity problem, or, more appropriate to this discussion, a dislocation. Dundurs and Santare provide a concise mathematical discussion of the use of a dislocation as the required singularity in an otherwise undisturbed elastic body, which provides Green's function, which leads to the stress field solution of the boundary value problem in plane elasticity (*8*).

What is a Green's function[1]? To answer that question, first we must understand the Dirac delta function. The Dirac delta function is used to describe a sudden, rapid rise in force, described by a forcing function. So, for example, if we have an undisturbed piece of plywood lying on the floor and we place a nail somewhere in the middle but do not yet strike the nail with a hammer, then the plywood is at a state of zero stress everywhere. Suddenly, we strike the nail with a hammer so that the plywood is at a state of zero stress everywhere except under the tip of the nail where the stress rapidly approaches some huge value (call it a conceptual infinity). The forcing function describes the force applied by the nail to the plane of plywood, and the Dirac delta function describes the change in condition from zero everywhere to infinity under the tip of the nail.

The Dirac delta function has these properties:

$$\delta\left(x-\xi\right)=0, \;\; x\neq\xi \; , \tag{12}$$

$$\int_{\xi-\beta}^{\xi+\beta}\delta\left(x-\xi\right)dx=1, \;\; \beta>0 \; , \tag{13}$$

$$\int_{\xi-\beta}^{\xi+\beta}f(x)\delta\left(x-\xi\right)dx=f(\xi), \;\; \beta>0 \; . \tag{14}$$

Mathematically, to solve a differential equation containing the Dirac delta function, we need to transform the Dirac delta function. We use the third property, equation 14, to perform a transformation using a linear differential operator, $\mathcal{L}$, so that

[1]Apparently, there is some debate as to whether it is correct to say "Green function" or "Green's function". A somewhat recent article showed that the occurrence of the use of "Green's function" has taken precedence in recent years (*10*). Since Green's functions are problem specific and more idea than identity, I prefer to use "Green's function".

$$\mathcal{L}g(x,\xi) = \delta(x-\xi) \; . \tag{15}$$

According to Stakgold (*9*), as explained by Dundurs and Santare (*8*), any nonnegative, locally integrable function $f(x)$ for which

$$\int_R f(x)dx = 1 \; , \tag{16}$$

and such a function $f(x)$ containing β, where $\beta > 0$,

$$f_\beta(x) = \frac{1}{\beta} f\left(\frac{x}{\beta}\right) \; , \tag{17}$$

approaches the Dirac delta function, $\delta(x)$ as β approaches zero:

$$\lim_{\beta\to 0} f_\beta(x) = \delta(x) \; . \tag{18}$$

This technique is known as $mollification$, where the function $f_\beta(x)$ is a mollifier or "averaging kernel" and any function that is convoluted with that function has its sharp features smoothed (it is mollified). The technique is used here to provide that a linear differential operation on the function $f(x)$ produces the delta function in accordance with equation 15, and as such, may be thought of as requiring the solution for the concentrated unit of inhomogeneity to be a distribution of stresses around the singularity rather than an infinite stress at the singularity. Thus, the singularity is smoothed or mollified.

The Dirac delta function has the properties of being zero everywhere except at the point of some singularity, say, at point $x = \xi$, where the Dirac delta function suddenly (conceptually) approaches infinity. Then the integral—over any domain, say R, containing that point of singularity, $x = \xi$, as long as that point is not one of the end points of the domain—of any function, such as equation 17, with these properties of the Dirac delta function will equal one as stated by equation 16.

In mathematics, the Green's function is defined as the solution to an inhomogeneous differential equation (e.g., the boundary value problem of an infinite elastic medium containing a dislocation) that is subject to a concentrated unit impulse (e.g., the delta function representing the concentrated

unit of inhomogeneity—the dislocation) at a prescribed point $\xi = (\xi, \eta) \in \mathbf{\Omega}$ inside the domain (e.g., the location of the dislocation in an infinite elastic medium). The Green's function solution is the linear superposition (e.g., using an integral in a continuum to accumulate the stress disturbances due to the dislocation in the elastic medium concentrated at position ξ) to sum up Green's function solutions to (homogeneous) individual unit impulse problems. The Green's function is the solution to the homogeneous portion of the differential equation everywhere $x \neq \xi$. Then the solution to the boundary value problem, which is "forced" and so a non-homogeneous differential equation (e.g., not to confuse with the term "inhomogeneity" used above; here we mean a differential equation not equal to zero, $p(x)\frac{d^2u}{dx^2} + q(x)\frac{du}{dx} + r(x)u(x) = f(x)$, where $f(x)$ is the forcing function), is a linear superposition of the Green's function solutions to the individual unit impulse problems (*11*). Then, once the Green's function is known, the solution to the forced boundary value problem is found from a convolution integral of the forcing function (itself a superposition of delta impulses) and the Green's function, for example, in two dimensions

$$u(x,y) = \int\int_{\Omega} G(x,y;\xi,\eta) f(\xi,\eta) d\xi d\eta \; . \tag{19}$$

Further, in terms of dislocations, taking the derivative with respect to x of the Green's function at the point of concentrated inhomogeneity $g(x,\xi)$, i.e., $\frac{\partial g}{\partial x}$ (a linear differential operation), will result in a function with the properties described by equations 16, 17, and 18. That is, the derivative of the Green's function must have a jump discontinuity at the impulse point $x = \xi$. So we can use the singularity of a jump (discontinuity) in displacement seen across a dislocation as a forcing function because it has these properties, and so it is the solution to the boundary value problem of the otherwise undisturbed plane containing a dislocation. The Green's function then characterizes how the system responds to the forcing function and that, in terms of mechanics, is the stress field of the system. See Dundurs and Santare (*8*) for further discussion on using a dislocation as the singularity that provides a Green's function that gives the stress field solution to the problem.

3.3 Dislocations and Cracks

Further, Dundurs and Santare (*8*) point out the similarity of a crack to a series of dislocations. Then, if we know the solution for a single dislocation, this opens the possibility of using a superposition of that solution for the solution of a pile-up of dislocations. If we know the solution for a pile-up of dislocations, which is similar mathematically to a crack, that is, a singular disturbance in an elastic body, then we can use this solution as the Green's function solution to the plane elasticity boundary value problem of interest. Bilby and Eshelby (*12*), in 1968, were

the first to discuss the use of a series of dislocations as the solution for the stress field of a crack in an elastic medium. The following discussion is based on the discussion by Dundurs and Santare (*8*), which is, in turn, based on the discussion by Bilby and Eshelby (*12*).

Consider an infinite linear elastic medium, which contains a continuous line of dislocations along the x-axis, $-a < x < a$, as seen in figure 7. If we were to magnify our view of the pile-up of dislocations, it might look something like figure 8. In figure 8, we see that pairs of each of three identical edge dislocations adds an incremental step in displacement of thickness $\mathbf{b}$ to a region (the crack). If we add more and more dislocation pairs and make the dislocations smaller and smaller in thickness $\mathbf{b}$, then in the limit we approach a continuous distribution of dislocations along the x-axis that may be represented by some function $B(x)$.

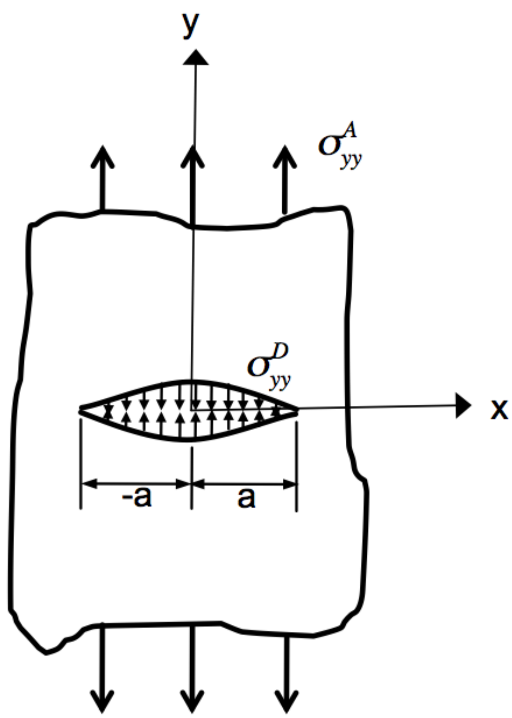

Figure 7. Infinite elastic body, loaded at infinity and containing a crack of length 2a.

We are interested in knowing the stress field around a crack. The crack lies in an infinite elastic medium (figure 7) and we know the size of the crack ($-a < x < a$) and we know the elastic material properties: Poisson's ratio ν, shear modulus μ, and the relationships in equation 20.

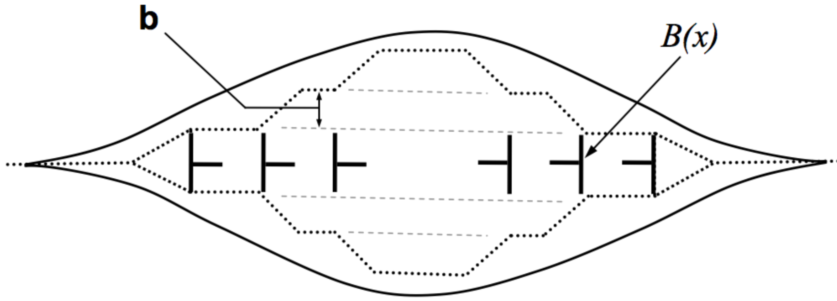

Figure 8. Schematic interpretation, from Dundurs and Santare (*8*), of what a crack represented by dislocations might look like.

$$\kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \frac{(3-\nu)}{(1+\nu)} & \text{plane stress} \end{cases} \tag{20}$$

The crack is represented by a series of dislocation pairs having a distribution that is given by a forcing function $B(x,y)$. Then, to determine the stress field around the crack in the elastic medium, we simply add up all the little bits of stress caused by all the little bits of deformation caused by all the little dislocations described by $B(x,y)$ with the point of singularity of interest (i.e., the crack tip or dislocation) located at (ξ, η) . That is,

$$\sigma_{yy}^{D}(x,y) = \int B(\xi,\eta)\frac{1}{b}\sigma_{yy}(x,y;\xi,\eta)d\xi d\eta \;, \tag{21}$$

and the stress components in the other directions are similar, for example, with σ_{xx} or σ_{xy} replacing σ_{yy}. In equation 21, we have an integral equation of the Green's function solution, $G = \frac{1}{b}\sigma_{yy}(x,y;\xi,\eta)$, for a single edge dislocation located at point $(x,y) = (\xi,\eta)$, with the forcing function $B(x)$.

So if we consider the continuous distribution of dislocations to lie on the x-axis, where $y = 0$ so that the dislocation distribution may be written as $B(x)$, and consider the dislocation distribution to lie between $-a < x < a$ and have a Burgers vector in the y direction, then we can see that at some location between x and $x + dx$ anywhere along this line, the net change in dislocations is given by $db = B(x)dx$. This simplifies equation 21 to

$$\sigma_{yy}^{D}(x,y) = \frac{2\mu}{\pi(\kappa+1)}\int_{-a}^{a}\frac{B(\xi)}{x-\xi}d\xi \;, \quad -a < x < a \;. \tag{22}$$

The preceding discussion is true for Mode I crack opening (i.e., tension) and the stress components in the other directions are found in a similar manner. We could also find the stress components for the other crack opening modes, e.g., Mode II (i.e., anti-plane shear) using screw dislocations, and similarly Mode III, using appropriate orientations in each case for the dislocation distribution and corresponding Burgers vector.

Similar to the discussion of inclusions above, if we observe that figure 8 represents a pile-up of tiny steps in displacement, then if we insert the accumulation of those jumps in displacement (i.e., the crack) into an undisturbed elastic medium, we would cause some state of stress in the elastic medium. Thus, for our collection of dislocations along the x-axis, where $-a < x < a$ and $y = 0$, tractions would be instigated in the otherwise undisturbed elastic medium. Thus, our now disturbed elastic medium would experience compressive tractions along the x-axis, where $-a < x < a$ and $y = 0$ due to the inserted accumulation of displacements, and it would experience tensile tractions everywhere else along the x-axis, that is, $|x| > a$.

In fracture mechanics, we typically impose boundary conditions requiring that the crack faces be traction free (*13, 14*). But we just realized that there is a compressive traction in the region of the crack due to the inserted continuous distribution of dislocations, and we found that equation 22 describes the stress in the region due to this insertion. If we realize that we stated earlier that the infinite elastic medium was undisturbed such that there are no forces applied at infinity, then we can see that if we were to apply some stress at infinity, say, σ^A_{yy}, we could counteract the stress due to the crack, σ^D_{yy}, by ensuring that $\sigma^A_{yy} + \sigma^D_{yy} = 0$. Then if we were to remove the pile-up of tiny steps in displacement, which we know are described by the function $B(x)$, then that same space would be empty, like a crack, but it is still described by the function $B(x)$ and the stress field around it is still described by σ^D_{yy}. Thus, we have described the crack in terms of the tiny steps in displacement caused by a pile-up of dislocations.

Typically, we know or can measure the value of the forces applied at infinity, σ^A_{yy}, but we do not know the function, $B(x, y)$, that describes the continuous distribution of dislocations (a.k.a., the crack). So if we have a specimen containing a straight crack of size 2a and we measure the forces applied at infinity, then we let the coordinate axes lie on the crack such that the crack lies along the x-axis with the center at $y = 0$ and $-a < x < a$, as in figure 7, then we can find the unknown function $B(x, 0)$ using equation 22 set equal to the negative of the tractions applied at infinity, σ^A_{yy}. Then if we consider the point of singularity, the crack tip, which lies at $(x, y) = (\xi, \eta)$, we have

$$\frac{2\mu}{\pi(\kappa+1)}\int_{-a}^{a}\frac{B(\xi)}{x-\xi}d\xi = -\sigma^A_{yy}\ , \quad -a < x < a\ , \tag{23}$$

but we need another constraint to solve this equation. If we recognize that where the crack closes there is no step in displacement, then we have the crack closure condition,

$$\int_{-a}^{a} B(\xi)d\xi = 0 \,. \quad -a < x < a \tag{24}$$

The solution to equations 23 and 24 is well known and is given by

$$B(x) = -\sigma_{yy}^{A} \frac{(\kappa+1)}{2\mu} \frac{x}{\sqrt{a^2 - x^2}} \,, \quad -a < x < a \,, \tag{25}$$

$$B(x) = 0 \,, \quad |x| > a \,, \tag{26}$$

Equations 25 and 26 simply state that the function that describes the continuous distribution of dislocations is equal to equation 25 on the crack, but it is equal to zero away from the crack where there is no displacement due to dislocations.

Now we know the function $B(x, 0)$ that will result in zero tractions on the crack surface. That is, $\sigma_{yy}(x, 0) = 0$ on $-a < x < a$. Likewise, due to the tractions at infinity, the stresses in the specimen everywhere else on x, along $y = 0$, away from the crack, will be given by

$$\sigma_{yy}(x, 0) = \frac{\sigma_{yy}^{A}\,|x|}{\sqrt{x^2 - a^2}} \,. \tag{27}$$

In similar fashion, solutions may be found for Mode II and Mode III cracks as described by Dundurs and Santare (*8*). Also, now we have the solution for an infinite elastic matrix containing a crack, which is represented by dislocations. The solution for the interaction of a crack with an inclusion then may be found by using the solution from the interaction of an edge dislocation with an inclusion as the Green's function in the problem of a crack interacting with an inclusion in an infinite elastic medium.

3.4 Green's Function for a Dislocation

In an infinite elastic body with no tractions applied at the boundaries (at infinity), for a single edge dislocation located within this body, the elastic stress field in the body due to this dislocation is given by the following (Cartesian) stress components, σ_{ij},

$$\sigma_{xx}(x,y,\xi,\eta) = \frac{2\mu b_y}{\pi(\kappa+1)} \frac{(x-\xi)[(x-\xi)^2-(y-\eta)^2]}{r^4} \;, \tag{28}$$

$$\sigma_{yy}(x,y,\xi,\eta) = \frac{2\mu b_y}{\pi(\kappa+1)} \frac{(x-\xi)[(x-\xi)^2+3(y-\eta)^2]}{r^4} \;, \tag{29}$$

$$\sigma_{xy}(x,y,\xi,\eta) = \frac{2\mu b_y}{\pi(\kappa+1)} \frac{(y-\eta)[(x-\xi)^2-(y-\eta)^2]}{r^4} \;, \tag{30}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2$, b_y is the Burgers vector oriented in the y direction, μ is shear modulus, and κ is a relationship to Poisson's ratio as described in equation 20 (*8*). The stresses may be found for any arbitrary location in the Cartesian plane, say, a distance of r from the origin, at an angle of θ from the x-axis, using a simple transformation of axes, for example (*15*),

$$\begin{aligned}
\sigma^2_{xx} &= \sigma^1_{xx}\cos^2\theta + \sigma^1_{yy}\sin^2\theta - 2\sigma^1_{xy}\sin\theta\cos\theta \;, \\
\sigma^2_{yy} &= \sigma^1_{xx}\sin^2\theta + \sigma^1_{yy}\cos^2\theta + 2\sigma^1_{xy}\sin\theta\cos\theta \;, \\
\sigma^2_{xy} &= \left(\sigma^1_{xx} - \sigma^1_{yy}\right)\sin\theta\cos\theta + \sigma^1_{xy}\left(\cos^2\theta - \sin^2\theta\right) \;.
\end{aligned} \tag{31}$$

So if we let the edge dislocation be at the origin of the coordinate axes, that is, $(\xi,\eta) = (0,0)$, and then if, like a crack along the x-axis at $y = 0$, we look closer and closer at the dislocation (which is oriented in the y-direction) so that in the limit the dislocation approaches the x-axis as $y \to 0$, then from equation 29 we have

$$
\begin{aligned}
\sigma_{yy}(x,y,0,0) &= \frac{2\mu b_y}{\pi(\kappa+1)} \frac{(x-0)[(x-0)^2+3(y-0)^2]}{\left((x-0)^2+(y-0)^2\right)^2} \\
&= \left(\frac{A}{\pi}\right) \frac{x(x^2+3y^2)}{\left(x^2+y^2\right)^2} \\
&= \left(\frac{A}{\pi}\right) \frac{x}{(x^2+y^2)} \left(1+\frac{2y^2}{(x^2+y^2)}\right) \\
&= \left(\frac{A}{\pi}\right) \left[\left(\frac{x}{(x^2+y^2)}\right)\left(\frac{1/x^2}{1/x^2}\right)\right] \left(1+\frac{2y^2}{(x^2+y^2)}\right) \\
&= \left(\frac{A}{\pi}\right) \left[\frac{1/x}{\left(1+(y/x)^2\right)}\right] \left(1+\frac{2y^2}{(x^2+y^2)}\right) \qquad (32) \\
\therefore \sigma_{yy} &\propto \frac{1}{\pi} \frac{1/x}{\left(1+(y/x)^2\right)}
\end{aligned}
$$

where $A = \frac{2\mu b_y}{(\kappa+1)}$ is a constant dependent on material properties and is introduced here only for convenience. Taking the limit of σ_{yy} as $y \to 0$ we see that

$$
\begin{aligned}
\lim_{y\to 0} \sigma_{yy} &\propto \frac{1}{\pi} \frac{1/x}{\left(1+(y/x)^2\right)} \\
\lim_{y\to 0} \sigma_{yy} &\propto \frac{1}{x} \, ,
\end{aligned}
$$

so the stress is singular (that is, a singularity or a point force) at the point of the dislocation. Also if we let $x = \beta$, where β is any number, then take the limit of σ_{yy} as $\beta \to \infty$ we see that

$$
\begin{aligned}
\sigma_{yy}(x,y)\Big|_{x=\beta} &\propto \frac{1}{\pi} \frac{1/\beta}{\left(1+(y/\beta)^2\right)} \\
\sigma_{yy}(x,y)\Big|_{x=\beta} &\propto \frac{1}{\pi} \frac{\beta}{(1+\beta^2 y^2)} \, , \qquad (33)
\end{aligned}
$$

so that

$$\lim_{\beta\to\infty} \sigma_{yy}^{\beta}(y) = \begin{cases} 0\,, & y \neq 0 \\ \infty\,, & y = 0 \end{cases} \quad , \tag{34}$$

and

$$\int_{-\infty}^{\infty} \sigma_{yy}^{\beta}(y)dy = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{\beta}{(1+\beta^2y^2)}dy = \frac{1}{\pi}\tan^{-1}\beta y\Bigg|_{y=-\infty}^{\infty} = \frac{1}{\pi}\pi = 1 \tag{35}$$

Notice that equations 34 and 35 satisfy the requirements for the delta function given by equations 12 and 13 (note that equation 13 is the same requirement as given by equation 16). So we can conclude that σ_{yy} satisfies the requirements for the delta function.

Also notice that if we evaluate the limit in equation 34 as $\beta \to 0$ we see that

$$\sigma_{yy}\Bigg|_{x\to 0^+} \propto \delta(y)\,, \tag{36}$$

which is the same as concluded by equation 18. Similar arguments may be made for the other stress components. So the σ_{ij} satisfy the requirements for a delta function and so may be used as the Green's function in integral equations.

Therefore, we now know that the Green's function $G(x, y, \xi, \eta)$ as in equation 19 is given by σ_{ij} for a dislocation interacting with a boundary, and σ_{ij} satisfies the requirements of a delta function so that we can use the properties of equation 14 to solve equation 21 for the unknown forcing function $B(x)$. The forcing function, call it $B(x)$ in equation 21 or call it $f(\xi, \eta)$ in equation 19, or whatever, is known for the problem of the stress field in an infinite elastic medium caused by the presence of a crack, which is represented as a pile-up of dislocations as discussed previously. But now we know how to find the forcing function for the problem of the stress field in an infinite elastic medium caused by the presence of a crack near an inclusion. All we need are the Green's functions for the integral equations and we can solve the integral equations for the forcing functions. We know σ_{ij} can be used as the Green's functions, so we use equations 28, 29, and 30. Using these stress solutions as our Green's functions, we can find the unknown forcing functions. When we have the forcing functions, we can solve for the stress field around the crack and inclusion. From the stresses, we can determine the stress intensity factors near the crack and the inclusion, which tell us how the crack will behave and are important for predicting fracture. Also

once we know the stresses, we can describe the strains by a constitutive model such as Hooke's law. We can then also find the displacements by integrating the strains.

4. Solving the Crack-Inclusion Interaction

The first published solution to the crack-inclusion interaction problem is attributed to Tamate in 1968 (*16*). Tamate provided the solution for the effect of a circular elastic inclusion on a crack. He found the stress field around a crack and an inclusion mathematically using the complex stress potential method; a famous example of this method is the Westergaard solution for the stress field around a crack in an infinite elastic medium (*13*). Sanford provides a review of solving mechanics problems using complex potentials (*14*). Tamate followed Muskhelishvili's method for the solution of complex potentials using a power series expansion to determine a set of simultaneous equations to determine the coefficients in the power series (*17*). In his example, Tamate solved 20 equations in 20 unknowns. A major shortcoming of the Tamate solution is that the solution only works when the radius of the inclusion is much less than the distance from the center of the inclusion to the crack, and then only when that distance (center inclusion to the crack) was much less than that distance plus the crack length, that is, for relatively small inclusions near to relatively long cracks. This shortcoming was pointed out and addressed by Atkinson in his solution for the crack-inclusion problem (*18*). However, the Tamate paper, in addition to being the first solution, provided a critical finding for the fracture mechanics of the crack-inclusion interaction: an inclusion *more rigid* than the surrounding matrix will *decrease* the stress intensity factor in the neighborhood of the crack tip, and likewise an inclusion *less rigid* than the surrounding matrix will *increase* the stress intensity factor in the neighborhood of the crack tip (*8, 16*).

Atkinson in 1972 was the first to consider the crack as a continuous distribution of dislocations to develop a singular integral equation. He then solved the integral equation by two independent methods: first, he reduced the integral equation to a Fredholm equation and solved the Fredholm equation numerically by a method Atkinson published in a separate paper (*18*). This is the approach taken in this report. The second method used an expansion of Tschebyscheff polynomials (*18*). In setting up the integral equations, Atkinson was also the first to use the Dundurs and Mura solution for an edge dislocation interacting with a circular inclusion as the Green's function for the solution of a crack interacting with an inclusion (*7, 18*).

The Atkinson solution considered only cracks and dislocation pile-ups that lie on the x-axis

(i.e., where $y = 0$), which is a commonly used simplification in fracture mechanics (*13, 14*). However, shortly after, in 1974, Erdogan et al. (*19*) used a similar approach, that is, the Dundurs and Mura solution as a Green's function in a set of singular Fredholm integral equations, to solve the more general crack-inclusion interaction for an arbitrarily oriented crack. The technique used by Erdogan et al. has since become the most widely accepted and used method in the solution of various crack-inclusion problems; Dundurs and Santare provide a much more comprehensive review of the literature in this field (*8*).

4.1 Forming the Integral Equations

An integral equation includes an unknown function $\phi(t)$ that we want to find and it is related through an integral of a known kernel and the unknown function. If both limits are prescribed, we call the equation a Fredholm integral equation (if one is variable, it is called a Volterra integral equation) (*20*). For example, a Fredholm integral equation of the first kind is

$$f(x) = \int_a^b K(x,t)\phi(t)dt \tag{37}$$

where the function $f(x)$ and the integral kernel $K(x,t)$ are known and we want to determine the function $\phi(t)$. Similarly, a Fredholm integral equation of the second kind is

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)dt \tag{38}$$

where λ is some known constant (usually based on the elastic constants of our materials) and, as before, the function $f(x)$ and the integral kernel $K(x,t)$ are known and we want to determine the function $\phi(t)$, but now we also have a function $\phi(x)$. The function $\phi(x)$ is often found using a Neumann series (*20*).

We know from the earlier discussions (see sections 3.2 and 3.4) that the stress field solution for an edge dislocation, equations 28–30, has the properties of the delta function after being operated on by a linear differential operator and so may be used as a Green's function according to equation 15. We know that a crack may be represented mathematically as a continuous distribution of dislocations, and we know that we can determine the stress field of a crack interacting with an inclusion if we have the Green's function (see section 3.3). So we can use the stress field solution as the Green's function in the integral equations that describe the stress field solution for a crack interacting with an inclusion. So starting simply, with equation 37, we let the

kernel $K(x, t)$ equal the Green's function for an edge dislocation given by equations 28–30.

Presumably, we also know the size of the inclusion and the size of the crack, so as we build our integral equations, we set the limits of integration to the limits of inclusion size and crack size. Presumably also, we know the stress field in our infinite elastic medium. We may find that we have other functions in our integral equations and if we bring constants out in front of the integral, for example, as λ, we may find that our integral equation looks more like equation 38. Finding the solution to our integral equations will involve numerical methods.

4.2 Inclusion and Crack Geometry

Solutions have been found for inclusions that are flat, circular, or elliptical, and cracks may be straight or arc and cracks may be present in the matrix or the inclusion or both. Although a general solution is provided by Erdogan et al. for an arc crack (*19*), they necessarily simplify to a straight crack to avoid complicating coordinate system transformations. For simplicity, here we consider only a single straight crack in an infinite elastic medium. Further, we assume that the crack and inclusion are about the same order of magnitude. We work in two dimensions and assume that inclusions are "sparsley" located since, in the words of Erdogan et al., "The exact elasticity treatment of the three-dimensional problem with a regular or random array of elastic inclusions embedded into an elastic matrix containing an internal crack appears to be hopelessly complicated. (*19*)" Other simplifying assumptions that we make include the use of plane strain or generalized plane stress, and we require a perfect bond between the inclusion and the matrix. Finally, we consider a flat inclusion. The geometry we have just described for the problem of determining the stress field near a crack and inclusion in an infinite elastic medium is shown in figure 9. Here, we have chosen to use the notation provided by Erdogan and Wei (*21*). Note that our inclusion is specified by $h_o(x)$, but that since we are assuming a flat inclusion we require h_o to be a constant thickness for the inclusion and h_o must be small relative to the length of the inclusion $2a_1$. However, we could specify a function to describe the inclusion as a circle or an ellipse; for example, if we assume the size $2a_1$ is fixed for the inclusion (so that a and b are constant limits for our integral equations), then we could specify a function, say $h_o(x) = b_o\sqrt{1 - x^2}$, to provide an elliptical inclusion. Finally, as an "infinite" elastic medium, we assume that the boundaries of the medium are sufficiently far away from the region of the crack-inclusion interaction that the boundaries do not experience the stress influence from the crack or inclusion.

In figure 9 we have a flat inclusion of size $2a_1$ with limits a and b and constant height h_o and which lies on the x_1, y_1 coordinate axes. We also have a straight crack of size $2a_2$ with limits c

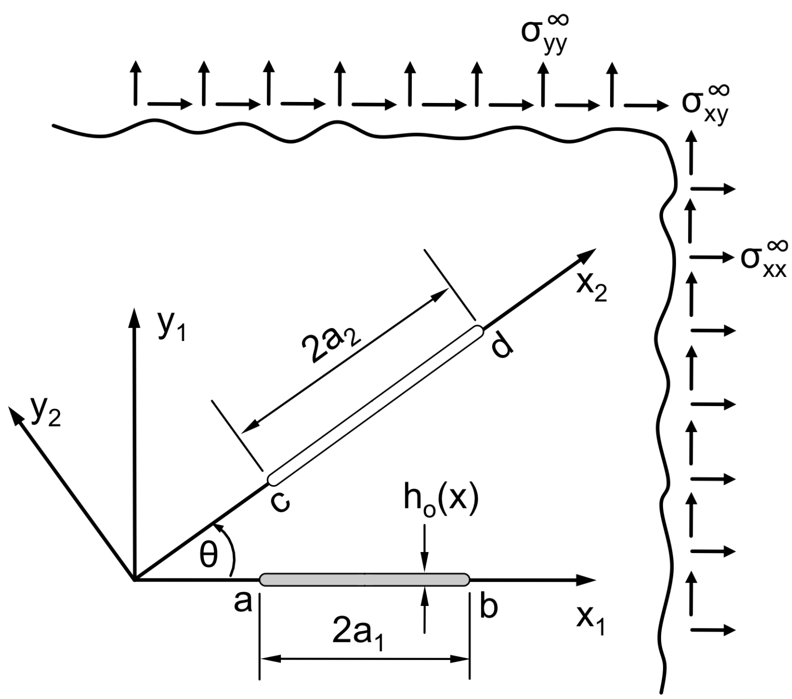

Figure 9. Geometry for the crack-inclusion problem, adapted from Erdogan and Wei (*21*) .

and d that lies on the x_2, y_2 coordinate axes. The x_2, y_2 coordinate axes are oriented relative to the x_1, y_1 coordinate axes by a rotation of angle θ. Far from the crack-inclusion interaction region, at the boundaries of the infinite elastic medium, we have applied tractions producing Cartesian stresses σ_{ij}^{∞} as shown in figure 9.

4.3 Displacement Derivatives

We have defined the geometry and the boundary conditions of our problem. We can assume that we know the elastic constants μ and ν (and from ν we have κ, given by equation 20) for the matrix and inclusion (or void, where $\mu = 0$). We have identified the Green's function for our integral equations. Now we need to identify the forcing function to complete our integral equations.

Using similar logic to the earlier discussions, if we have an undisturbed matrix and we insert an inclusion or we insert a crack, we would be inserting volume that would have a displacing effect on the surrounding matrix. So we use the jumps in displacement across the inclusion and across the crack as our concentrated units of inhomogeneity. For u_i and v_i, which are, respectively, the x_i and y_i components of displacement for $i = 1, 2$, the coordinate systems shown in figure 9. Then we take the derivative of these jumps in displacement and define our unknown functions:

$$g_1(x_1) = \frac{\partial}{\partial x_1}\left[v_1(x_1, 0^+) - v_1(x_1, 0^-)\right] \ , \ (a < x_1 < b) \ , \tag{39}$$

$$h_1(x_1) = \frac{\partial}{\partial x_1}\left[u_1(x_1, 0^+) - u_1(x_1, 0^-)\right] \ , \ (a < x_1 < b) \ , \tag{40}$$

$$g_2(x_2) = \frac{\partial}{\partial x_2}\left[v_2(x_2, 0^+) - v_2(x_2, 0^-)\right] \ , \ (c < x_1 < d) \ , \tag{41}$$

$$h_2(x_2) = \frac{\partial}{\partial x_2}\left[u_2(x_2, 0^+) - u_2(x_2, 0^-)\right] \ , \ (c < x_1 < d) \ . \tag{42}$$

We want to know what the stress field in the matrix near the crack and inclusion is. Referring to figure 9, we now have an unknown function g_1, which is the derivative of the discontinuity in the displacement of the matrix in the y_1 direction across the inclusion where $y_1 = 0$ and $(a < x_1 < b)$. Likewise, we have an unknown function h_1, which is the derivative of the discontinuity in the displacement of the matrix in the x_1 direction along the inclusion where $y_1 = 0$ and $(a < x_1 < b)$. We also have an unknown function g_2, which is the derivative of the discontinuity in the displacement of the matrix in the y_2 direction across the crack where $y_2 = 0$ and $(c < x_2 < d)$. Likewise, we have an unknown function h_2, which is the derivative of the discontinuity in the displacement of the matrix in the x_2 direction along the crack where $y_2 = 0$ and $(c < x_2 < d)$.

4.4 Stresses in the Inclusion

Considering boundary conditions for a moment, we know crack faces are traction free so we realize that the stresses must be zero inside the crack, but the displacement jump cannot be zero. Likewise, outside the crack, the stresses are not zero, but the displacement jump is zero. On the other hand, the inclusion is not free, rather we assumed it is perfectly bonded to the matrix, and let us suppose that the inclusion has some known elastic constants (i.e., it is not void), say, E_o, ν_o, μ_o. Then we must realize that the stresses both inside and outside of the inclusion will not be zero, and likewise the jumps in displacement will not be zero either inside or outside of the inclusion.

Let us assume that the variation of stress along the length of the inclusion is negligible and likewise the strain along the length of the inclusion is negligible. These are valid assumptions because the thickness of the inclusion is small relative to its length and because the length of the inclusion will only change by a negligible amount under stress. Let us also consider plane strain conditions here so that for the elastic constants we have

$$E_o = 2\mu_0(1+\nu_o)\ , \kappa_o = 3 - 4\nu_o\ . \tag{43}$$

We know from elementary elasticity theory (*15*) that strain is equal to the derivative of displacement or even more basically that strain is equal to the change in length divided by the original length. That is, in two dimensions, the strains in the x and y directions are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}\ ,\ \ \varepsilon_{yy} = \frac{\partial v}{\partial y}\ ,\ \ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\ , \tag{44}$$

or written another way,

$$\varepsilon_k = \frac{\Delta L_k}{L_k}\ , \tag{45}$$

where $k = x, y$ are the x and y directions.

Then let us observe that the change in the length of the inclusion, ΔL_k is the same as the jump in displacement in a given direction (i.e., $k = x, y$), and also observe that the original length in the y direction is $h_o(x)$. Recall that we assumed that the change in length, ∂u, is negligible along $a < x_1 < b$ so that we know from equation 44 that the strain in the inclusion in the x direction is $\varepsilon_{xx}^{inc} = 0$. Likewise, since $\partial u = 0$, the component of the shear strain, ε_{xy} from equation 44 that is dependent on ∂u, is also zero.

Therefore, we have for the strains in the inclusion

$$\varepsilon_{yy}^{inc}(x_1) \cong \frac{[v_1(x_1,0^+) - v_1(x_1,0^-)]}{h_o(x_1)}\ , \tag{46}$$

$$2\varepsilon_{xy}^{inc}(x_1) \cong \frac{[u_1(x_1,0^+) - u_1(x_1,0^-)]}{h_o(x_1)}\ . \tag{47}$$

Also, from equations 39 and 40, we have the partial derivatives

$$\partial\left[v_1(x_1,0^+) - v_1(x_1,0^-)\right] = g_1(x_1)\partial x_1\ , \tag{48}$$

$$\partial\left[u_1(x_1,0^+)-u_1(x_1,0^-)\right]=h_1(x_1)\partial x_1\ , \tag{49}$$

and if we integrate these along the inclusion we have an expression for the jumps in displacement:

$$\left[v_1(x_1,0^+)-v_1(x_1,0^-)\right]=\int g_1(x_1)dx_1\ , \tag{50}$$

$$\left[u_1(x_1,0^+)-u_1(x_1,0^-)\right]=\int h_1(x_1)dx_1\ . \tag{51}$$

Hooke's Law tells us that stress is related to strain by way of the elastic constants for the material (*15*), so we can write

$$\varepsilon_{yy}^{inc}(x_1)=\frac{1-\nu_o-2{\nu_o}^2}{E_o(1-\nu_o)}\sigma_{yy}^{inc}(x_1)\ , \tag{52}$$

$$\varepsilon_{yy}^{inc}(x_1)=\frac{1}{2\mu_o}\sigma_{xy}^{inc}(x_1)\ , \tag{53}$$

and if we solve for the stresses,

$$\sigma_{yy}^{inc}(x_1)=\frac{E_o(1-\nu_o)}{1-\nu_o-2{\nu_o}^2}\varepsilon_{yy}^{inc}(x_1)\ , \tag{54}$$

$$\sigma_{xy}^{inc}(x_1)=2\mu_o\varepsilon_{yy}^{inc}(x_1)\ . \tag{55}$$

Then if we plug equations 50 and 51 into equations 46 and 47 and then plug the result into equations 54 and 55 and finally convert the elastic constants using equation 43 (for plane strain), we find an expression for the stresses in the inclusion as a function of known elastic constants and our unknown displacement derivatives defined in equations 39 and 40:

$$\sigma_{yy}^{inc}(x_1)=\frac{\kappa_o+1}{\kappa_o-1}\frac{\mu_o}{h_o(x_1)}\int_a^{x_1}g_1(t)dt\ , \tag{56}$$

$$\sigma_{xy}^{inc}(x_1) = \frac{\mu_o}{h_o(x_1)} \int_a^{x_1} h_1(t)dt \ . \tag{57}$$

4.5 Stresses in the Matrix

We have found expressions for the (in plane) stresses in the inclusion as a function of known elastic constants and unknown displacement derivatives. Next, we need to find expressions for the stress disturbance in the matrix due to the crack and due to the inclusion. We now call upon the dislocation solution from Dundurs and Mura (*7*) (or similar). Earlier discussion demonstrated that we can use the dislocation solution as the kernel in a set of integral equations to determine the stresses in the plane due to the inclusion and the crack. Then with our unknown forcing functions defined earlier, the displacement derivatives of equations 39–42, we have the components of a Fredholm integral equation of the first kind.

The in plane stresses in the matrix are caused by the presence of the inclusion, that is, caused by the jumps in displacement due to the presence of the inclusion, that is, caused by the forcing functions defined in equations 39 and 40, the displacement derivatives. These equations are the $\phi(t)$ functions in our integral equations 37. The dislocation solution given by equations 28–30 characterizes the response of the matrix to the forcing functions from the inclusion. So the stresses at a point (x_1, y_1), referred to the x_1, y_1 coordinate system, due to the inclusion of size $a < x_1 < b$, may be expressed by

$$\sigma_{xx}^{11}(x_1, y_1) = \int_a^b \left[G_{xx}(x_1, y_1, t)g_1(t) + H_{xx}(x_1, y_1, t)h_1(t)\right] dt \ , \tag{58}$$

$$\sigma_{yy}^{11}(x_1, y_1) = \int_a^b \left[G_{yy}(x_1, y_1, t)g_1(t) + H_{yy}(x_1, y_1, t)h_1(t)\right] dt \ , \tag{59}$$

$$\sigma_{xy}^{11}(x_1, y_1) = \int_a^b \left[G_{xy}(x_1, y_1, t)g_1(t) + H_{xy}(x_1, y_1, t)h_1(t)\right] dt \ . \tag{60}$$

The stresses at a point (x_2, y_2), referred to the x_2, y_2 coordinate system, due to the crack of size $c < x_2 < d$, may be expressed by

$$\sigma_{xx}^{22}(x_2, y_2) = \int_c^d \left[G_{xx}(x_2, y_2, t)g_2(t) + H_{xx}(x_2, y_2, t)h_2(t)\right] dt \ , \tag{61}$$

$$\sigma_{yy}^{22}(x_2, y_2) = \int_c^d [G_{yy}(x_2, y_2, t)g_2(t) + H_{yy}(x_2, y_2, t)h_2(t)]\, dt \; , \tag{62}$$

$$\sigma_{xy}^{22}(x_2, y_2) = \int_c^d [G_{xy}(x_2, y_2, t)g_2(t) + H_{xy}(x_2, y_2, t)h_2(t)]\, dt \; . \tag{63}$$

In the tensor notation used here for stress, σ_{ij}^{kl}, we have $i, j = x, y$ representing the Cartesian stress components in the usual sense, and $k, l = 1, 2$, where $k = 1, 2$ indicates either stress caused by the inclusion $k = 1$ or stress caused by the crack $k = 2$ and $l = 1, 2$ indicates stresses referred to either coordinate system, x_1, y_1 for $l = 1$ or x_2, y_2 for $l = 2$. Also note that in earlier notation we used the point $(x, y) = (\xi, \eta)$ in our integral equations, but here we use the notation provided by Erdogan and Wei (*21*) so that the point $(x, y) = (\xi, \eta) = (t, 0)$, where since we assumed a flat inclusion, we have $y_1 = 0$ and on a crack we have $y_2 = 0$.

In equations 58–63, the Green's functions given by the dislocation solution provided in equations 28–30 (for a dislocation oriented in the y direction) must be rotated for edge dislocations in the appropriate x or y direction, that is, G_{ij} is for an edge dislocation oriented in the y direction and H_{ij} is for an edge dislocation oriented in the x direction. Note that x, y refer to the appropriate coordinate system, either x_1, y_1 or x_2, y_2, and μ and κ are elastic constants of the matrix where κ is given by equation 20. Then we have the following Green's functions:

$$G_{xx}(x, y, t, 0) = \left[\frac{2\mu}{(1+\kappa)}\right] \frac{(t-x)[(t-x)^2 - y^2]}{[(t-x)^2 + y^2]^2} \; , \tag{64}$$

$$G_{yy}(x, y, t, 0) = \left[\frac{2\mu}{(1+\kappa)}\right] \frac{(t-x)[(t-x)^2 + 3y^2]}{[(t-x)^2 + y^2]^2} \; , \tag{65}$$

$$G_{xy}(x, y, t, 0) = \left[\frac{2\mu}{(1+\kappa)}\right] \frac{(y)[-(t-x)^2 + y^2]}{[(t-x)^2 + y^2]^2} \; , \tag{66}$$

$$H_{xx}(x, y, t, 0) = \left[\frac{2\mu}{(1+\kappa)}\right] \frac{(y)[y^2 + 3(t-x)^2]}{[(t-x)^2 + y^2]^2} \; , \tag{67}$$

$$H_{yy}(x, y, t, 0) = \left[\frac{2\mu}{(1+\kappa)}\right] \frac{(y)[y^2 - (t-x)^2]}{[(t-x)^2 + y^2]^2} \; , \tag{68}$$

$$H_{xy}(x,y,t,0) = \left[\frac{2\mu}{(1+\kappa)}\right] \frac{(t-x)[-y^2+(t-x)^2]}{[(t-x)^2+y^2]^2} \ . \tag{69}$$

4.6 Coupling Stresses Between Crack and Inclusion

In addition to having an influence on the stress field in the matrix referred to the respective coordinate systems on which the inclusion or crack lies, the inclusion influences the stress in the matrix around the crack and the crack influences the stress in the matrix around the inclusion. These are coupling stresses between the inclusion and the crack. The coupling stresses are found using a transformation of coordinate axes as in equation 31, where the angle θ relates the two coordinate axes as in figure 9.

Then using the notation described above for σ_{ij}^{kl}, the coupling stresses in the matrix may be found from the previously defined stresses using the following transformations. Here the stresses from equations 58–60, which are calculated at point (x_1, y_1) above, are calculated at the point $(x_1, y_1) = (x_2\cos\theta, x_2\sin\theta)$ in the coupling stresses below, and likewise the stresses from equations 61–63, which are calculated at point (x_2, y_2) above, are calculated at the point $(x_2, y_2) = (x_1\cos\theta, -x_1\sin\theta)$ in the coupling stresses below.

Recall that we assumed the variation in stress in the x direction is negligible. The stress transformations for coupling stresses are the following:

$$\sigma_{yy}^{12}(x_1,0) = \sigma_{xx}^{22}(x_2,y_2)\sin^2\theta + \sigma_{yy}^{22}(x_2,y_2)\cos^2\theta + 2\sigma_{xy}^{22}(x_2,y_2)\sin\theta\cos\theta \ , \tag{70}$$

$$\sigma_{xy}^{12}(x_1,0) = \left(\sigma_{xx}^{22}(x_2,y_2) - \sigma_{yy}^{22}(x_2,y_2)\right)\sin\theta\cos\theta + \sigma_{xy}^{22}(x_2,y_2)\left(\cos^2\theta - \sin^2\theta\right) \ , \tag{71}$$

$$\sigma_{yy}^{21}(x_2,0) = \sigma_{xx}^{11}(x_1,y_1)\sin^2\theta + \sigma_{yy}^{11}(x_1,y_1)\cos^2\theta + 2\sigma_{xy}^{11}(x_1,y_1)\sin\theta\cos\theta \ , \tag{72}$$

$$\sigma_{xy}^{21}(x_2,0) = \left(\sigma_{xx}^{11}(x_1,y_1) - \sigma_{yy}^{11}(x_1,y_1)\right)\sin\theta\cos\theta + \sigma_{xy}^{11}(x_1,y_1)\left(\cos^2\theta - \sin^2\theta\right) \ . \tag{73}$$

Likewise, the Green's function transformations for coupling stresses are the following.

Plug the expressions from equations 58–63 into equations 70–73, ignore for now the integral signs and the displacement derivatives (they will return later), and collect and separate G and H terms to find the Green's function transformations for coupling stresses as follows:

$$G^{12}_{yy}(x_1, t) = G_{xx}(x_2, y_2)\sin^2\theta + G_{yy}(x_2, y_2)\cos^2\theta + 2G_{xy}(x_2, y_2)\sin\theta\cos\theta \ , \tag{74}$$

$$G^{12}_{xy}(x_1, t) = (G_{xx}(x_2, y_2) - G_{yy}(x_2, y_2))\sin\theta\cos\theta + G_{xy}(x_2, y_2)\left(\cos^2\theta - \sin^2\theta\right) \ , \tag{75}$$

$$H^{12}_{yy}(x_1, t) = H_{xx}(x_2, y_2)\sin^2\theta + H_{yy}(x_2, y_2)\cos^2\theta + 2H_{xy}(x_2, y_2)\sin\theta\cos\theta \ , \tag{76}$$

$$H^{12}_{xy}(x_1, t) = (H_{xx}(x_2, y_2) - H_{yy}(x_2, y_2))\sin\theta\cos\theta + H_{xy}(x_2, y_2)\left(\cos^2\theta - \sin^2\theta\right) \ , \tag{77}$$

where $(x_2, y_2) = (x_1\cos\theta, -x_1\sin\theta)$, and

$$G^{21}_{yy}(x_2, t) = G_{xx}(x_1, y_1)\sin^2\theta + G_{yy}(x_1, y_1)\cos^2\theta - 2G_{xy}(x_1, y_1)\sin\theta\cos\theta \ , \tag{78}$$

$$G^{21}_{xy}(x_2, t) = (G_{xx}(x_1, y_1) - G_{yy}(x_1, y_1))\sin\theta\cos\theta + G_{xy}(x_1, y_1)\left(\cos^2\theta - \sin^2\theta\right) \ , \tag{79}$$

$$H^{21}_{yy}(x_2, t) = H_{xx}(x_1, y_1)\sin^2\theta + H_{yy}(x_1, y_1)\cos^2\theta - 2H_{xy}(x_1, y_1)\sin\theta\cos\theta \ , \tag{80}$$

$$H^{21}_{xy}(x_2, t) = (H_{xx}(x_1, y_1) - H_{yy}(x_1, y_1))\sin\theta\cos\theta + H_{xy}(x_1, y_1)\left(\cos^2\theta - \sin^2\theta\right) \ , \tag{81}$$

where $(x_1, y_1) = (x_2 \cos\theta, x_2 \sin\theta)$.

Then the normal and shear components of the coupling stresses on the $y_1 = 0$ plane of the inclusion, which are due to the influence of the displacement derivatives $g_2(x_2)$ and $h_2(x_2)$ for the crack are

$$\sigma_{yy}^{12}(x_1, 0) = \int_c^d \left[G_{yy}^{12}(x_1, t) g_2(t) + H_{yy}^{12}(x_1, t) h_2(t)\right] dt \, , \tag{82}$$

$$\sigma_{xy}^{12}(x_1, 0) = \int_c^d \left[G_{xy}^{12}(x_1, t) g_2(t) + H_{xy}^{12}(x_1, t) h_2(t)\right] dt \, , \tag{83}$$

and the normal and shear components of the coupling stresses on the $y_2 = 0$ plane of the crack, which are due to the influence of the displacement derivatives $g_1(x_1)$ and $h_1(x_1)$ for the inclusion are

$$\sigma_{yy}^{21}(x_2, 0) = \int_a^b \left[G_{yy}^{21}(x_2, t) g_1(t) + H_{yy}^{21}(x_2, t) h_1(t)\right] dt \, , \tag{84}$$

$$\sigma_{xy}^{21}(x_2, 0) = \int_a^b \left[G_{xy}^{21}(x_2, t) g_1(t) + H_{xy}^{21}(x_2, t) h_1(t)\right] dt \, . \tag{85}$$

4.7 Boundary Conditions

Let the infinite elastic medium be uniformly loaded far away from the region of the crack and inclusion, as shown in figure 9. If the x_1, y_1 coordinate axes are aligned with the direction of the σ_{xx}^{∞} and σ_{yy}^{∞} stress components as shown in figure 9, and recalling again that stress difference in the x direction was assumed negligible, then we have the following stresses along the x_1 axis in the inclusion:

$$\sigma_{yy}^{1\infty}(x_1, 0) = \sigma_{yy}^{\infty} \, , \tag{86}$$

$$\sigma_{xy}^{1\infty}(x_1, 0) = \sigma_{xy}^{\infty} \, . \tag{87}$$

For coordinate axes that are not aligned with the directions of the uniform loading at infinity, such as the x_2, y_2 coordinate axes, a coordinate transformation as performed earlier and as is common in elementary elasticity (*15*) is needed. For an angle θ between the direction of the applied loads at infinity and the coordinate axes, as shown in figure 9, a transformation is performed according to equation 31. Then we have the following stresses along the x_2 axis in the crack:

$$\sigma_{yy}^{2\infty}(x_2, 0) = \sigma_{xx}^{\infty} \sin^2\theta + \sigma_{yy}^{\infty} \cos^2\theta - 2\sigma_{xy}^{\infty} \sin\theta \cos\theta \ , \tag{88}$$

$$\sigma_{xy}^{2\infty}(x_2, 0) = \left(\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty}\right) \sin\theta \cos\theta + \sigma_{xy}^{\infty} \left(\cos^2\theta - \sin^2\theta\right) \ . \tag{89}$$

We now have expressions for all of the stresses in the inclusion and in the matrix around the inclusion and the crack including (1) the stresses in the inclusion caused by the jump in displacement from the presence of the inclusion, which pushes on the matrix and causes the matrix to push back on the inclusion inducing stress within the inclusion; (2) the stresses in the matrix due to the jumps in displacement caused by the presence of the inclusion and the crack; (3) the stress caused in the crack by the jump in displacement of the inclusion, which pushes on the matrix and causes the matrix to push on the nearby crack, and the stress caused in the inclusion by the jump in displacement of the crack, which pushes on the matrix and causes the matrix to push on the nearby inclusion; and (4) the stresses caused in the crack and in the inclusion by the tractions applied at the far boundaries of the infinite elastic matrix.

If we recognize that for equilibrium, since the inclusion is assumed to be perfectly bonded to the surrounding matrix, the stresses in the inclusion must equal the stresses in the surrounding matrix, then we can sum the stresses in the matrix around the inclusion and set them equal to the stresses in the inclusion. Then we have the following boundary conditions in the inclusion where $y_1 = 0$ and $a < x_1 < b$:

$$\sigma_{yy}^{11}(x_1, 0) + \sigma_{yy}^{12}(x_1, 0) + \sigma_{yy}^{1\infty}(x_1, 0) = \sigma_{yy}^{inc}(x_1) \ , \tag{90}$$

$$\sigma_{xy}^{11}(x_1, 0) + \sigma_{xy}^{12}(x_1, 0) + \sigma_{xy}^{1\infty}(x_1, 0) = \sigma_{xy}^{inc}(x_1) \ . \tag{91}$$

We know that crack faces are traction free (*14*), so inside the crack there must be zero stress, and so if we sum the stresses in and around the crack, they must cancel to zero so that there is no

stress inside the crack. Then we have the following boundary conditions in the crack where $y_2 = 0$ and $c < x_2 < d$,

$$\sigma_{yy}^{22}(x_2, 0) + \sigma_{yy}^{21}(x_2, 0) + \sigma_{yy}^{2\infty}(x_2, 0) = 0 \; , \tag{92}$$

$$\sigma_{xy}^{22}(x_2, 0) + \sigma_{xy}^{21}(x_1, 0) + \sigma_{xy}^{2\infty}(x_2, 0) = 0 \; . \tag{93}$$

We have integral expressions for each of the components of equations 90–93 in terms of known elastic constants, known Green's functions, and unknown displacement derivatives. Now we can construct a set of integral equations and solve for the unknown functions.

4.8 Integral Equations for the Crack-Inclusion Interaction

By plugging the stress expressions given by equations 56 and 57, 59–63, 82–85, and 86–89 into the boundary conditions given by equations 90–93, we obtain the following set of integral equations for the crack-inclusion problem.

Along the inclusion, where $y_1 = 0$ and $a < x_1 < b$, we have

$$\begin{aligned}\int_a^b \left[G_{yy}(x_1,t)g_1(t) + H_{yy}(x_1,t)h_1(t)\right] dt + \int_c^d \left[G_{yy}^{12}(x_1,t)g_2(t) + H_{yy}^{12}(x_1,t)h_2(t)\right] dt + \sigma_{yy}^{\infty} \\ = \int_a^{x_1} \frac{\kappa_o + 1}{\kappa_o - 1}\frac{\mu_o}{h_o(x_1)} g_1(t)dt \; ,\end{aligned} \tag{94}$$

$$\begin{aligned}\int_a^b \left[G_{xy}(x_1,t)g_1(t) + H_{xy}(x_1,t)h_1(t)\right] dt + \int_c^d \left[G_{xy}^{12}(x_1,t)g_2(t) + H_{xy}^{12}(x_1,t)h_2(t)\right] dt + \sigma_{xy}^{\infty} \\ = \int_a^{x_1} \frac{\mu_o}{h_o(x_1)} h_1(t)dt \; .\end{aligned} \tag{95}$$

Along the crack, where $y_2 = 0$ and $c < x_2 < d$,

$$\int_c^d \left[G_{yy}(x_2,t)g_2(t) + H_{yy}(x_2,t)h_2(t)\right]dt + \int_a^b \left[G_{yy}^{21}(x_2,t)g_1(t) + H_{yy}^{21}(x_2,t)h_1(t)\right]dt \\ + \sigma_{xx}^{\infty}\sin^2\theta + \sigma_{yy}^{\infty}\cos^2\theta - 2\sigma_{xy}^{\infty}\sin\theta\cos\theta = 0\ , \tag{96}$$

$$\int_c^d \left[G_{xy}(x_2,t)g_2(t) + H_{xy}(x_2,t)h_2(t)\right]dt + \int_a^b \left[G_{xy}^{21}(x_2,t)g_1(t) + H_{xy}^{21}(x_2,t)h_1(t)\right]dt \\ + \left(\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty}\right)\sin\theta\cos\theta + \sigma_{xy}^{\infty}\left(\cos^2\theta - \sin^2\theta\right) = 0\ . \tag{97}$$

These equations may be simplified further if we plug the equations 64–69 into the integral equations 94–97 where appropriate and evaluate the relations for equations 64–69 at $(x_1, y_1 = x_1, 0, t, 0)$ in integrals evaluated along the inclusion $a < x_1 < b$ and evaluate the relations for equations 64–69 at $(x_2, y_2 = x_2, 0, t, 0)$ in integrals evaluated along the crack $c < x_2 < d$.

Then from equations 65 and 68 we get

$$\begin{aligned}
G_{yy}(x,y,t,0) &= \left[\frac{2\mu}{(1+\kappa)}\right]\frac{(t-x)[(t-x)^2+3y^2]}{[(t-x)^2+y^2]^2} \\
G_{yy}(x_1,0,t,0) &= \left[\frac{2\mu}{(1+\kappa)}\right]\frac{(t-x_1)[(t-x_1)^2+3(0)^2]}{[(t-x_1)^2+(0)^2]^2} \\
G_{yy}(x_1,t) &= \left[\frac{2\mu}{(1+\kappa)}\right]\frac{(t-x_1)^3]}{(t-x_1)^4} = \left[\frac{2\mu}{(1+\kappa)}\right]\frac{1}{t-x_1}\ ,
\end{aligned} \tag{98}$$

$$\begin{aligned}
H_{yy}(x,y,t,0) &= \left[\frac{2\mu}{(1+\kappa)}\right]\frac{(y)[y^2-(t-x)^2]}{[(t-x)^2+y^2]^2} \\
H_{yy}(x_1,0,t,0) &= \left[\frac{2\mu}{(1+\kappa)}\right]\frac{(0)[0^2-(t-x_1)^2]}{[(t-x_1)^2+(0)^2]^2} \\
H_{yy}(x_1,0,t,0) &= 0\ ,
\end{aligned} \tag{99}$$

and plugging these into equation 94 and simplifying, we get

$$
\begin{aligned}
\frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_a^b \frac{g_1(t)}{t-x_1}dt - \frac{\mu_o(\kappa_o+1)}{\kappa_o-1}\int_a^{x_1}\frac{g_1(t)}{h_o(x_1)}dt \\
+\int_c^d G_{yy}^{12}(x_1,t)g_2(t)dt + \int_c^d H_{yy}^{12}(x_1,t)h_2(t)dt = -\sigma_{yy}^{\infty}\,.
\end{aligned}
\tag{100}
$$

Equation 100 may be further expanded by plugging in equations 74 and 76, which are, in turn, expanded by plugging in equations 64–69. Then equation 101 results.

This expression has become quite cumbersome so we do not expand the remaining integral equations completely in this fashion; this expansion is conducted here only to show the complete integral equation for the normal stresses in the inclusion. In similar fashion, we may expand and simplify equations 95–97. Thus we have the integral equations for the crack-inclusion interaction problem. Observe that, once the geometry, as in figure 10, and material properties of the problem are fully defined, the integral equations are composed entirely of $known$ quantities save the four $unknown$ displacement derivatives, equations 39–42. So we have four integral equations in four unknowns. We can solve for the four unknowns and then completely describe the stress field of the crack-inclusion interaction. However, it is obvious that solving these four integral equations in four unknowns is only practical using numerical methods.

$$
\begin{aligned}
&\frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_a^b \frac{g_1(t)}{t-x_1}dt - \frac{\mu_o(\kappa_o+1)}{\kappa_o-1}\int_a^{x_1}\frac{g_1(t)}{h_o(x_1)}dt \\
&\quad+\int_c^d \frac{2\mu}{1+\kappa}\frac{(t-x_1\cos\theta)[(t-x_1\cos\theta)^2-(-x_1\sin\theta)^2]}{[(t-x_1\cos\theta)^2+(-x_1\sin\theta)^2]^2}\sin^2\theta\; g_2(t)dt \\
&\quad+\int_c^d \frac{2\mu}{1+\kappa}\frac{(t-x_1\cos\theta)[(t-x_1\cos\theta)^2+3(-x_1\sin\theta)^2]}{[(t-x_1\cos\theta)^2+(-x_1\sin\theta)^2]^2}\cos^2\theta\; g_2(t)dt \\
&\quad+\int_c^d \frac{2\mu}{1+\kappa}\frac{(-x_1\sin\theta)[-(t-x_1\cos\theta)^2+(-x_1\sin\theta)^2]}{[(t-x_1\cos\theta)^2+(-x_1\sin\theta)^2]^2}\sin\theta\cos\theta\; g_2(t)dt \\
&\quad+\int_c^d \frac{2\mu}{1+\kappa}\frac{(-x_1\sin\theta)[(-x_1\sin\theta)^2+3(t-x_1\cos\theta)^2]}{[(t-x_1\cos\theta)^2+(-x_1\sin\theta)^2]^2}\sin^2\theta\; h_2(t)dt \\
&\quad+\int_c^d \frac{2\mu}{1+\kappa}\frac{(-x_1\sin\theta)[(-x_1\sin\theta)^2-(t-x_1\cos\theta)^2]}{[(t-x_1\cos\theta)^2+(-x_1\sin\theta)^2]^2}\cos^2\theta\; h_2(t)dt \\
&\quad+\int_c^d \frac{2\mu}{1+\kappa}\frac{(t-x_1\cos\theta)[-(-x_1\sin\theta)^2+(t-x_1\cos\theta)^2]}{[(t-x_1\cos\theta)^2+(-x_1\sin\theta)^2]^2}\sin\theta\cos\theta\; h_2(t)dt = -\sigma_{yy}^{\infty}\,.
\end{aligned}
\tag{101}
$$

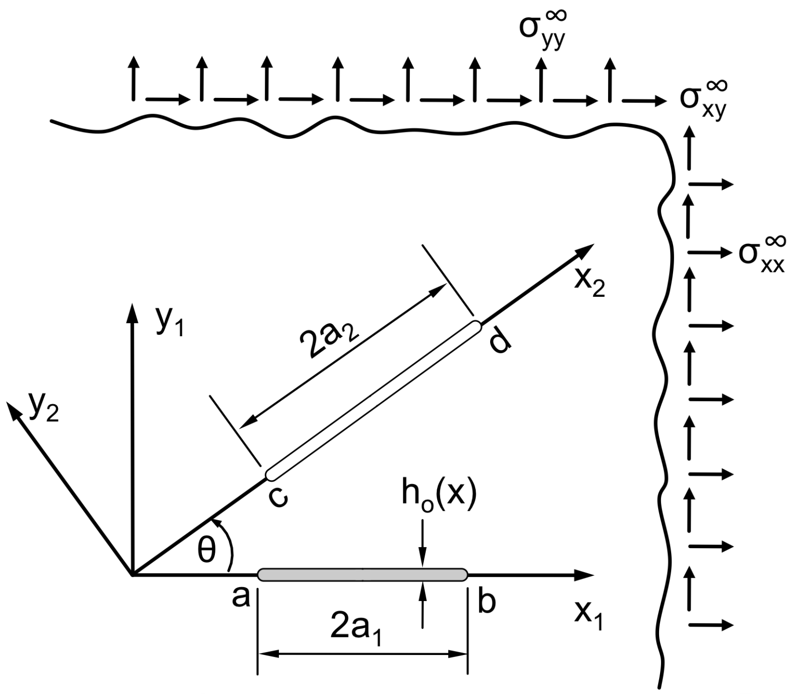

Figure 10. The crack-inclusion problem geometry, adapted from Erdogan and Wei (*21*): figure 9 is repeated here in the text for convenience.

5. Solution to the Crack-Inclusion Interaction Problem

Referring to figure 10, the solution to the interaction between an elliptical inclusion and an arbitrarily oriented straight crack is given by solving these four equations 102–105 in four unknowns so that we have on the inclusion $(a < x_1 < b)$,

$$\frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_a^b \frac{g_1(t)}{t-x_1}dt - \mu_o\frac{\kappa_o+1}{\kappa_o-1}\int_a^{x_1}\frac{g_1(t)}{h_o(x_1)}dt + \int_c^d G_{yy}^{12}(x_1,t)g_2(t)dt + \int_c^d H_{yy}^{12}(x_1,t)h_2(t)dt = -\sigma_{yy}^{\infty}\,, \tag{102}$$

$$
\begin{aligned}
&\frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_a^b \frac{h_1(t)}{t-x_1}dt - \mu_o\int_a^{x_1}\frac{h_1(t)}{h_o(x_1)}dt \\
&\qquad\qquad + \int_c^d G_{xy}^{12}(x_1,t)g_2(t)dt + \int_c^d H_{xy}^{12}(x_1,t)h_2(t)dt = -\sigma_{xy}^{\infty} \ ,
\end{aligned}
\tag{103}
$$

and on the crack $(c < x_2 < d)$,

$$
\begin{aligned}
&\int_a^b G_{yy}^{21}(x_2,t)g_1(t)dt + \int_a^b H_{yy}^{21}(x_2,t)h_1(t)dt \\
&+ \frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_c^d \frac{g_2(t)}{t-x_2}dt = -\sigma_{xx}^{\infty}\sin^2\theta - \sigma_{yy}^{\infty}\cos^2\theta + 2\sigma_{xy}^{\infty}\sin\theta\cos\theta \ ,
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
&\int_a^b G_{xy}^{21}(x_2,t)g_1(t)dt + \int_a^b H_{xy}^{21}(x_2,t)h_1(t)dt \\
&+ \frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_c^d \frac{h_2(t)}{t-x_2}dt = -\left(\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty}\right)\sin\theta\cos\theta - \sigma_{xy}^{\infty}\left(\cos^2\theta - \sin^2\theta\right) \ ,
\end{aligned}
\tag{105}
$$

where μ and ν are elastic constants of the matrix; μ_o and ν_o are elastic constants of the inclusion; κ and κ_o are given by equation 20 as a function of ν or ν_o, respectively; $h_o(x_1)$ is a function that describes the height of the inclusion as a function of x_1 along the length of the inclusion; the unknown functions g_1, h_1, g_2, and h_2 are given by equations 39–42; the Green's function transforms G_{ij}^{kl} and H_{ij}^{kl} are given by equations 74–81; and the Green's functions, G_{ij} and H_{ij} in equations 74–81 are given by equations 65–69.

5.1 Reduction to Well Known Solutions

If we let $g_2 = h_2 = 0$, this is the same as removing the crack from the medium so that we now have an infinite elastic medium containing an elliptical inclusion. With $g_2 = h_2 = 0$, there is no crack to influence the inclusion and the coupling stress terms of equations 102 and 103 disappear, and equations 104 and 105 become zero. Then we can remove the x_2, y_2 coordinate system from figure 10 and let the x_1, y_1 coordinate system be simply the x, y coordinate system and, as in figure 11, place the origin at the center of the inclusion and normalize so that on the inclusion $(-1 < x < 1)$, we have two equations in two unknowns

$$
\frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_{-1}^{1}\frac{g_1(t)}{t-x}dt - \mu_o\frac{\kappa_o+1}{\kappa_o-1}\int_{-1}^{x}\frac{g_1(t)}{h_o(x)}dt = -\sigma_{yy}^{\infty} \ ,
\tag{106}
$$

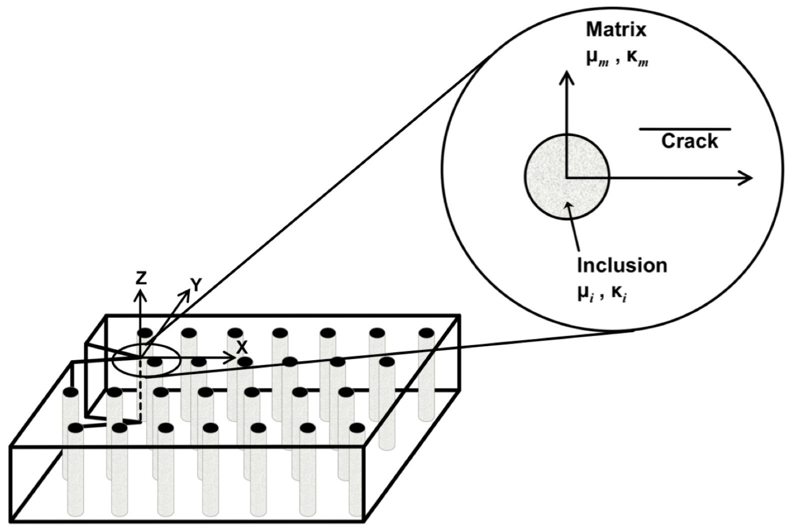

Figure 11. Example of crack-inclusion interaction: composites, reinforcing fiber in an elastic matrix. Adapted from Savalia (*22*).

$$\frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_{-1}^{1}\frac{h_1(t)}{t-x}dt-\mu_o\int_{-1}^{x}\frac{h_1(t)}{h_o(x)}dt=-\sigma_{xy}^{\infty}\ . \tag{107}$$

We can examine this solution further for an elliptical cross section inclusion. We define the inclusion height function $h_o(x)$ as

$$h_o(x)=c_0\sqrt{1-x^2}\ , \tag{108}$$

where c_0 is some scalar coefficient that describes the magnitude of the inclusion.

Then the solution becomes

$$\frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_{-1}^{1}\frac{g_1(t)}{t-x}dt-\mu_o\frac{\kappa_o+1}{\kappa_o-1}\int_{-1}^{x}\frac{g_1(t)}{c_0\sqrt{1-x^2}}dt=-\sigma_{yy}^{\infty}\ , \tag{109}$$

$$\frac{2\mu}{(1+\kappa)}\frac{1}{\pi}\int_{-1}^{1}\frac{h_1(t)}{t-x}dt-\mu_o\int_{-1}^{x}\frac{h_1(t)}{c_0\sqrt{1-x^2}}dt=-\sigma_{xy}^{\infty}\ . \tag{110}$$

If we simplify these expressions by filtering out the constants such that

$$c_1 = \frac{(\kappa + 1)}{2\mu} \ , \tag{111}$$

$$c_2 = \frac{\mu_o(\kappa + 1)(\kappa_o + 1)}{2\mu c_0(\kappa_o - 1)} \ , \tag{112}$$

$$c_3 = \frac{\mu_o(\kappa + 1)}{2\mu c_0} \ , \tag{113}$$

then we can write

$$\frac{1}{\pi}\int_{-1}^{1} \frac{g_1(t)}{t - x} dt - c_2 \int_{-1}^{x} \frac{g_1(t)}{\sqrt{1 - x^2}} dt = -c_1 \sigma_{yy}^{\infty} \ . \tag{114}$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{h_1(t)}{t - x} dt - c_3 \int_{-1}^{x} \frac{h_1(t)}{\sqrt{1 - x^2}} dt = -c_1 \sigma_{xy}^{\infty} \ . \tag{115}$$

Then the solutions to equations 114 and 115 are found to be

$$g_1(t) = -\left[\frac{c_1 \sigma_{yy}^{\infty}}{1 + c_2}\right] \frac{t}{\sqrt{1 - t^2}} \ , \ (-1 < t < 1) \ , \tag{116}$$

$$h_1(t) = -\left[\frac{c_1 \sigma_{xy}^{\infty}}{1 + c_3}\right] \frac{t}{\sqrt{1 - t^2}} \ , \ (-1 < t < 1) \ . \tag{117}$$

For known elastic constants and known loading at infinity, equations 116 and 117 may be used to determine the normal and shear stress fields caused by the presence of an elliptical inclusion in an infinite elastic medium.

There are two noteworthy special cases discussed by Erdogan and Wei (*21*). First, if the crack is removed as just discussed but the stiffness of the remaining inclusion is zero (i.e., $\mu_o = 0$), then we have mathematically the same situation as a crack. That is, an inclusion in an infinite elastic matrix, if the inclusion has zero stiffness, has the same boundary conditions as a crack in an infinite elastic matrix. This situation is also representative of a void in an infinite elastic matrix. In the case of an inclusion with zero stiffness, the solution reduces to the solution for a crack in linear elastic fracture mechanics, which is well known and may be found, for example, in a

fracture mechanics textbook (*13*). Second, if the crack is not removed but the stiffness of the inclusion is zero (i.e., $\mu_o = 0$), then as before the boundary conditions in the zero stiffness inclusion are the same as in a crack: that is, the stresses within the inclusion become zero. Then the terms in equations 102 and 103 derived from equations 56 and 57, which are the stresses in the inclusion, become zero. That is, the $\mu_o \int_a^{x_1}$ terms of equations 102 and 103 become zero, and the resulting solution given by equations 102–105 reduces to the solution for two arbitrarily oriented cracks.

5.2 Solution by Numerical Methods

Various authors have described solution of problems in fracture mechanics using the boundary integral equation method, for example, an early treatment by Tan and Fenner (*23*). These methods have also been extended to solution of the crack-inclusion interaction problem, for example, by Tan et al. (*24*). This report does not provide a detailed examination of the solution to such problems by numerical methods, but this section, adapted from Santare (*25*), is included as a very brief introduction to the solution method for the integral equations discussed above.

If we have a cracked elastic body and we take the origin of the coordinate axes at the center of the crack, as in figure 12, and then if we normalize by dividing all lengths by the half-crack length, a, we have something similar to the following Fredholm integral equation of the second kind:

$$\int_{-1}^{1} \frac{B(t)}{t-x} dt + \int_{-1}^{1} K(x,t) B(t) dt = f(x) \tag{118}$$

Observe that equation 118 is similar in form to the integral equations 102–105 discussed earlier. Here $B(t)$ represents unknown functions similar to $g_1(t)$, $h_1(t)$, $g_2(t)$, and $h_2(t)$. $K(x,t)$ is the kernel of the integral equation, the Green's functions for the problem similar to G_{ij} and H_{ij}, which will be different for each problem depending on geometry (singularity) and boundary conditions. $f(x)$ is some function dependent on boundary conditions, for example, in equations 102–105 the tractions applied at infinity and the stresses in the inclusion (which in equations 102–105 have been moved to the left hand side of the equation).

As in figure 12, we have a crack, which we know must contain a singularity. So we know the unknown function $B(t)$ must contain a singularity. To make the unknown function easier to determine, we can take out the singular portion. If we define a function $\phi(t)$, which is bounded (i.e., not singular) and well behaved for $-1 < t < 1$, then we have

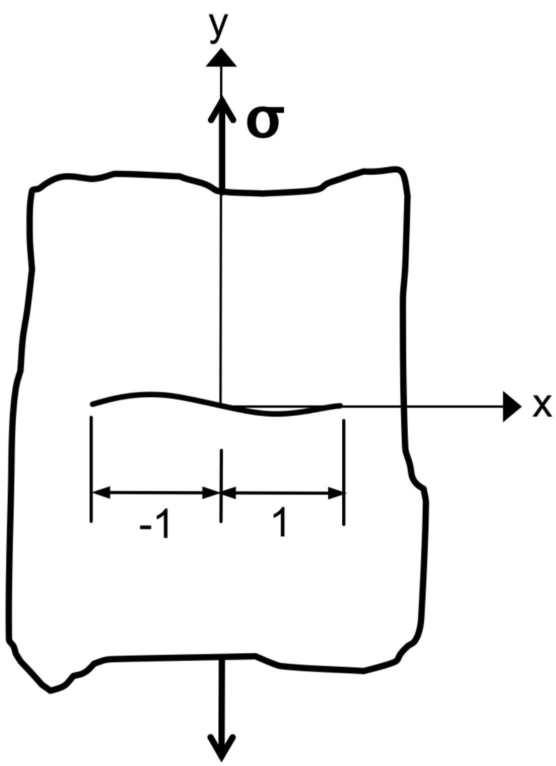

Figure 12. Cracked elastic body.

$$B(t) = \frac{\phi(t)}{\sqrt{1-t^2}}\ . \tag{119}$$

We know equation 119 has the necessary square root singularity of a crack tip at the crack tips, when $t = 1$ and when $t = -1$. Then by removing this singularity from the unknown function, it is easier to find $\phi(t)$ than it would have been to find $B(t)$.

To find $\phi(t)$, we begin by discretizing the crack into evenly spaced finite elements (or points in the one-dimensional case shown here), as in figure 13. The points t_{2k+i} are separated by a distance h. The points are then subdivided by points $\phi(t)$ in between the t_{2k+i} points. Now we can use a numerical interpolation method such as the Lagrangian interpolation function to interpolate the values of $\phi(t)$ between the t_{2k+i} points.

Lagrange interpolation uses parabolas to interpolate the curved line between the points as seen in figure 13. The first term of the Lagrangian interpolation function is

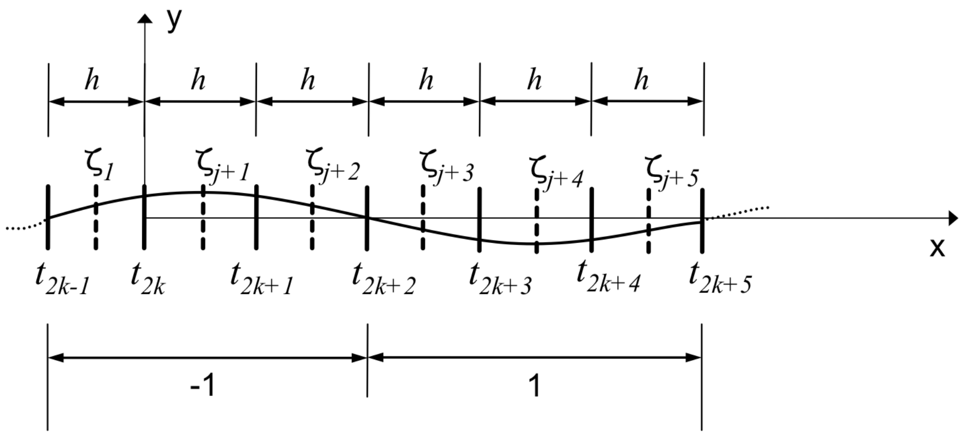

Figure 13. Discretized crack.

$$\phi(t) \approx \phi(t_{2k-1}) \frac{[(t - t_{2k})\,(t - t_{2k+1})]}{2h^2} \tag{120}$$

Adding more terms provides a more accurate interpolation. The second-order accurate Lagrange interpolation approximation then is given by

$$\begin{aligned}
\phi(t) &= \phi(t_{2k-1}) \frac{[(t - t_{2k})\,(t - t_{2k+1})]}{2h^2} && (121)\\
&- \phi(t_{2k}) \frac{[(t - t_{2k-1})\,(t - t_{2k+1})]}{2h^2} && (122)\\
&+ \phi(t_{2k+1}) \frac{[(t - t_{2k})\,(t - t_{2k-1})]}{2h^2} + \cdots && (123)
\end{aligned}$$

Presumably we know the value of the function at the boundaries (thus, the name, boundary integral), so we can take the known value of the unknown function ϕ at the first t point (e.g., at one boundary) and the value of ϕ at the last t point and interpolate for some number of t points between. Then take the interpolated values of $\phi(t)$ at known values of t and insert each $\phi(t)$ and corresponding t into equation 119 to find the value of $B(t)$ at that point t. Then for equation 119 we have $B(t)$ in terms of a set of unknown numerical values, $\phi(t_i)$, instead of a set of unknown functions $\phi(t)$. We can then plug these values of $B(t)$ into equation 118, the integral equation, then integrate. Integrating equation 118 we get an algebraic function $f(x)$ in terms of a set of unknown numbers instead of an integral equation in terms of unknown functions.

Then referring to figure 13, the solution is

$$\sum_{i=1}^{2N+1} \left(W_i(\xi_j) + V_i(\xi_j)K(\xi_j, t_i)\right) \phi(t_i) = f(\xi_j) \ , \ j = 1, 2, \ldots 2N \tag{124}$$

where W_i and V_i are weight functions determined from the assumed form of $B(x)$.

6. Conclusions

6.1 Stress Intensity Factor

According to Erdogan et al. (*19*), when the stress transformations are made before solving the integral equations, as in this report, the stress intensity factors may be determined as follows. Taking the standard definition of Modes I and II stress intensity factors given by equations 125 and 132, and if we have solved the integral equations 102–105 for the displacement derivatives g_1, h_1, g_2, and h_2 defined by equations 39–42, then we can determine the stress intensity factors using equations 133–140. Others, for example, Erdogan et al. (*19*) and Erdogan and Wei in (*21*), have solved the integral equations using the technique described by Erdogan (*26*), and from their results, they have determined the stress intensity factors for crack-inclusion interactions of various geometries and stiffness ratios between the shear modulus of the matrix (μ) and the shear modulus of the inclusion (μ_o). Their stress intensity factor results are tabulated and reported elsewhere (*19, 21*), so only qualitative results are discussed in this report.

Thus the stress intensity factors are given by the following definitions. At the a and b tips of the inclusion $(a < x_1 < b)$ in figures 9 and 10, the stress intensity factors are given by

$$K_I(a) = \lim_{x_1 \to a} \sqrt{2(a - x_1)} \, \sigma^1_{yy}(x_1, 0) \ , \tag{125}$$

$$K_{II}(a) = \lim_{x_1 \to a} \sqrt{2(a - x_1)} \, \sigma^1_{xy}(x_1, 0) \ , \tag{126}$$

$$K_I(b) = \lim_{x_1 \to b} \sqrt{2(b - x_1)} \, \sigma^1_{yy}(x_1, 0) \ , \tag{127}$$

$$K_{II}(b) = \lim_{x_1 \to b} \sqrt{2(b - x_1)}\, \sigma^1_{xy}(x_1, 0)\ . \tag{128}$$

At the c and d tips of the crack $(c < x_2 < d)$ in figures 9 and 10, the stress intensity factors are given by

$$K_I(c) = \lim_{x_2 \to c} \sqrt{2(c - x_2)}\, \sigma^2_{yy}(x_2, 0)\ , \tag{129}$$

$$K_{II}(c) = \lim_{x_2 \to c} \sqrt{2(c - x_2)}\, \sigma^2_{xy}(x_2, 0)\ , \tag{130}$$

$$K_I(d) = \lim_{x_2 \to d} \sqrt{2(d - x_2)}\, \sigma^2_{yy}(x_2, 0)\ , \tag{131}$$

$$K_{II}(d) = \lim_{x_2 \to d} \sqrt{2(d - x_2)}\, \sigma^2_{xy}(x_2, 0)\ . \tag{132}$$

Then the stresses, σ^k_{ij} (i,j=x, y, k=$1, 2$), needed to determine the stress intensity factors in equations 125–132 may be found by solving the integral equations 102–105 for the displacement derivatives g_1, h_1, g_2, and h_2 defined by equations 39–42. Then recognizing that the displacement derivatives (g_1, h_1, g_2, and h_2) along with the elastic constants (μ and κ) give the stress, we can substitute these into equations 125–132, then the stress intensity factors are given by the following equations 133–140.

At the a and b tips of the inclusion $(a < x_1 < b)$ in figures 9 and 10, the stress intensity factors are given by

$$K_I(a) = \frac{2\mu}{1+\kappa} \lim_{x_1 \to a} \sqrt{2(x_1 - a)}\, g_1(x_1)\ , \tag{133}$$

$$K_{II}(a) = \frac{2\mu}{1+\kappa} \lim_{x_1 \to a} \sqrt{2(x_1 - a)}\, h_1(x_1)\ , \tag{134}$$

$$K_I(b) = -\frac{2\mu}{1+\kappa} \lim_{x_1 \to b} \sqrt{2(b - x_1)}\, g_1(x_1)\ , \tag{135}$$

$$K_{II}(b) = -\frac{2\mu}{1+\kappa} \lim_{x_1 \to b} \sqrt{2(b - x_1)}\, h_1(x_1)\ . \tag{136}$$

At the c and d tips of the crack $(c < x_2 < d)$ in figures 9 and 10, the stress intensity factors are given by

$$K_I(c) = \frac{2\mu}{1+\kappa} \lim_{x_2 \to c} \sqrt{2(x_2 - c)}\, g_2(x_2)\ , \tag{137}$$

$$K_{II}(c) = \frac{2\mu}{1+\kappa} \lim_{x_2 \to c} \sqrt{2(x_2 - c)}\, h_2(x_2)\ , \tag{138}$$

$$K_I(d) = -\frac{2\mu}{1+\kappa} \lim_{x_2 \to d} \sqrt{2(d - x_2)}\, g_2(x_2)\ , \tag{139}$$

$$K_{II}(d) = -\frac{2\mu}{1+\kappa} \lim_{x_2 \to d} \sqrt{2(d - x_2)}\, h_2(x_2)\ . \tag{140}$$

6.2 Summary

Generally for a matrix with a crack tip near an inclusion, the crack will propagate toward the inclusion if the inclusion stiffness is less than the matrix stiffness (including zero stiffness for a void or another crack). Conversely, the crack will tend to propagate away from the inclusion if the inclusion stiffness is greater than the matrix stiffness. Figure 14 illustrates qualitatively how a crack will interact with an inclusion depending on the relative stiffness.

We found that the stress field solution for an edge dislocation has the properties of a delta function and so may be used as a Green's function. We also found that a crack may be represented as a continuous distribution of dislocations and if we know the Green's function, we can determine the stress field for the crack. So we used the stress field solution for an edge dislocation as the Green's function for a continuous distribution of dislocations, which models a crack in an infinite elastic medium. We also found that we can determine the stress field in and around an elastic inclusion in an infinite elastic medium if we have the Green's function. Then also we can use the same edge dislocation stress field solution as the Green's function to determine the stresses due to the presence of an inclusion. Using the Green's function for an edge dislocation as the kernel in a set of integral equations that describe the stress field in and around an inclusion and a crack, we

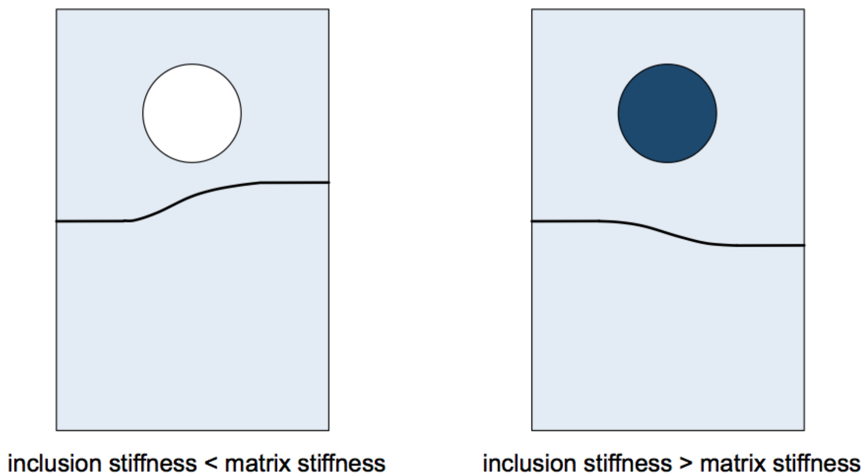

Figure 14. The interaction between a crack and an inclusion. Schematic inspired by Pais (*27*).

can see from the form of a Fredholm integral equation that we have a set of unknown functions that must be identified to solve for the stress field in an infinite elastic medium due to the presence of a crack and an inclusion. We found that we can use functions that describe continuous distributions of dislocations to represent a crack and an inclusion. Then we use these unknown distribution functions as our unknown functions in our integral equations. Defining the form of these unknown functions as the derivatives of the discontinuities in displacement due to the presence of a crack in an elastic medium and the discontinuities in displacement due to the presence of an inclusion in an elastic medium, we can assemble the integral equations. Finally, if we remove the singular portion of the unknown functions, to make them well behaved, we can discretize the crack into finite elements and interpolate between the known values of the unknown functions at the boundaries of the crack and inclusion and plug the results into the integral equations, which we then solve simultaneously to determine numerical values for the unknown functions in the integral equations. Then we have numerical values for the stress distribution in and around a crack and an inclusion and we can use these values to determine the stress intensity factors that result in an elastic medium due to the interaction between a crack and an inclusion.

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